Homework 9 Solutions

19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers. Use any theorems you wish. (a) $f(x) = x^{17}\sin x - e^x \cos 3x$ on $[0,\pi]$; (d) $f(x) = x^3$ on $\mathbb{R}$; (f) $f(x) = \sin\frac{1}{x^2}$ on $(0,1]$; (g) $f(x) = x^2 \sin\frac{1}{x}$ on $(0,1]$.

Solution. (a) Since x^17, $\sin x$, e^x, and $\cos(3x)$ are continuous on $\mathbb{R}$, f is continuous on $\mathbb{R}$, and so is continuous on $[0,\pi]$. Since $[0,\pi]$ is a bounded and continuous interval, and f is continuous on this interval, by Theorem 19.2, f is uniformly continuous on $[0,\pi]$.

(d) $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. To justify this claim, we find two sequences real numbers (x_n) and (y_n) such that $x_n - y_n \to 0$ but $f(x_n) - f(y_n) \not\to 0$. We define $x_n = n + \frac{1}{n}$ and $y_n = n$, $n \in \mathbb{N}$. We have $x_n - y_n = \frac{1}{n} \to 0$, but $f(x_n) - f(y_n) = (n+\frac{1}{n})^3 - n^3 = 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3n$, which implies that $f(x_n) - f(y_n) \to +\infty$. So we get $f(x_n) - f(y_n) \not\to 0$, as desired.

(f) f is not uniformly continuous on $(0,1]$. Suppose f is uniformly continuous on $(0,1]$. Then for any Cauchy sequence (x_n) in $(0,1]$, we have that $(f(x_n))$ is also Cauchy by Theorem 19.4. Now we define $x_n = (n\pi+\pi/2)^{-1/2}$, $n \in \mathbb{N}$. For $n \in \mathbb{N}$, since $n\pi+\pi/2 \geq 1$, we get $x_n \in (0,1]$. So (x_n) is a sequence in $(0,1]$. Since $n\pi + \pi/2 \to +\infty$, we have $(n\pi+\pi/2)^{-1} \to 0$, and so $x_n = \sqrt{(n\pi+\pi/2)^{-1}} \to 0$. Thus, (x_n) is a Cauchy sequence in $(0,1]$. We calculate $f(x_n) = \sin(x_n^{-2}) = \sin(n\pi + \frac{\pi}{2}) = (-1)^n$, $n \in \mathbb{N}$. We have studied that $(f(x_n))$ is not convergent, and so is not Cauchy. The contradiction shows that f is not uniformly continuous on $(0,1]$.

(g) f is uniformly continuous on $(0,1]$. We define $\widetilde{f}$ on $[0,1]$ such that $\widetilde{f}(x) = f(x)$ for $x \in (0,1]$, and $\widetilde{f}(0) = 0$. We claim that $\widetilde{f}$ is continuous on $[0,1]$. Since $\widetilde{f}$ agrees with f on $(0,1]$, and f is continuous on $(0,1]$, $\widetilde{f}$ is continuous at every $x \in (0,1]$. It remains to show that $\widetilde{f}$ is continuous at 0. We need to show that if (x_n) is a sequence in $(0,1]$ such that $x_n \to 0$, then $\widetilde{f}(x_n) \to \widetilde{f}(0)$, which is equivalent to that $f(x_n) \to 0$. From $x_n \to 0$, we get $x_n^2 \to 0$. Since $|\sin(1/x_n)| \leq 1$, we have $|f(x_n)| = |x_n^2 \sin(\frac{1}{x_n})| \leq |x_n|^2$. By squeeze lemma, we have $f(x_n) \to 0$, as desired. Thus, $\widetilde{f}$ is continuous on $[0,1]$. Thus, $\widetilde{f}$ is a continuous extension of f on $[0,1]$. By Theorem 19.5, f is uniformly continuously on $(0,1]$. $\square$

19.2 Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the $\varepsilon - \delta$ property in Definition 19.1. (c) $f(x) = \frac{1}{x}$ on $[\frac{1}{2},\infty)$.

Solution. We observe that for $x, y \in [\frac{1}{2},\infty)$,

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{|y-x|}{|xy|} = \frac{|x-y|}{xy} \leq 4|x-y|,$$

where the last inequality holds because $\frac{1}{xy} \leq \frac{1}{1/2 \cdot 1/2} = 4$.

For any $\varepsilon > 0$, set $\delta = \varepsilon/4 > 0$. Suppose $x, y \in [\frac{1}{2}, \infty)$ and $|x - y| < \delta$, then from the above displayed formula, $|f(x) - f(y)| \leq 4|x - y| < 4\delta = \varepsilon$. By Definition 19.1, f is uniformly continuous on $[\frac{1}{2}, \infty)$. $\square$

19.4 (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. Hint: Assume not. Use Theorems 11.5 and 19.4.

(b) Use (a) to give yet another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Proof. (a) Suppose f is not bounded on S. Then for any $n \in \mathbb{N}$, there is $x_n \in S$ such that $|f(x_n)| > n$. So we get $|f(x_n)| \to \infty$. Since S is bounded, (x_n) is a bounded sequence. By Bolzano-Weierstrass Theorem (Theorem 11.5), (x_n) contains a convergent subsequence (x_{n_k}), which is a Cauchy sequence. By Theorem 19.4, $(f(x_{n_k}))$ is also a Cauchy sequence. So $f(x_{n_k})$ converges. On the other hand, from $|f(x_n)| \to +\infty$, we get $|f(x_{n_k})| \to +\infty$, which contradicts that $f(x_{n_k})$ converges. Thus, f is bounded on S.

(b) If $\frac{1}{x^2}$ is uniformly continuous on $(0, 1)$, then since $(0, 1)$ is bounded, by (a) $\frac{1}{x^2}$ is bounded on $(0, 1)$. However, if we take $x_n = \frac{1}{n+1} \in (0, 1)$, $n \in \mathbb{N}$, then $\frac{1}{x_n^2} = (n+1)^2 \to \infty$, which contradicts the boundedness of $\frac{1}{x^2}$ on $(0, 1)$. So $\frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$. $\square$

19.5 Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems or Exercise 19.4(a). (a) $\tan x$ on $[0, \frac{\pi}{4}]$, (b) $\tan x$ on $[0, \frac{\pi}{2})$, (c) $\frac{1}{x} \sin^2 x$ on $(0, \pi]$, (d) $\frac{1}{x-3}$ on $(0, 3)$, (e) $\frac{1}{x-3}$ on $(3, \infty)$, (f) $\frac{1}{x-3}$ on $[4, \infty)$.

Solution. (a) We know that $\tan x = \frac{\sin x}{\cos x}$ is continuous on $\mathbb{R} \backslash \{x : \cos x = 0\} = \mathbb{R} \backslash \{n\pi + \frac{\pi}{2} : n \in \mathcal{Z}\}$. So $\tan x$ is continuous on $[0, \frac{\pi}{4}]$. Since $[0, \frac{\pi}{4}]$ is a bounded closed interval, $\tan x$ is uniformly continuous on $[0, \frac{\pi}{4}]$.

(b) If $\tan x$ is uniformly continuous on $[0, \frac{\pi}{2})$, then by Exercise 19.4, $\tan x$ is bounded on $[0, \frac{\pi}{2})$. However, as $x_n \in (0, \frac{\pi}{2})$ and $x_n \to \frac{\pi}{2}$, we have $\sin(x_n) \to \sin(\frac{\pi}{2}) = 1$, $\cos(x_n) \to \cos(\frac{\pi}{2}) = 0$, and $\cos(x_n) > 0$ for all n. Thus, $\tan(x_n) = \frac{\sin(x_n)}{\cos(x_n)} \to \infty$. This implies that $\tan x$ is not bounded on $[0, \frac{\pi}{2})$. The contradiction shows that $\tan x$ is not uniformly continuous on $[0, \frac{\pi}{2})$.

(c) From Example 9 of Section 19, $\frac{\sin x}{x}$ extends to a continuous function on $\mathbb{R}$. Since $\sin x$ is continuous on $\mathbb{R}$, $\frac{1}{x} \sin^2 x = \frac{\sin x}{x} \cdot \sin x$ also extends to a continuous function on $\mathbb{R}$. Thus, $\frac{1}{x} \sin^2 x$ restricted to $(0, \pi]$ has a continuous extension to $[0, \pi]$. By Theorem 19.5, $\frac{1}{x} \sin^2 x$ is uniformly continuous on $(0, \pi]$.

(d) If $\frac{1}{x-3}$ is uniformly continuous on $(0, 3)$, then by Exercise 19.4, $\frac{1}{x-3}$ is bounded on $(0, 3)$. However, if we choose $x_n = 3 - \frac{1}{n} \in (0, 3)$, $n \in \mathbb{N}$, then $\frac{1}{x_n - 3} = -n \to -\infty$. So $\frac{1}{x-3}$ is not bounded on $(0, 3)$. The contradiction shows that $\frac{1}{x-3}$ is not uniformly continuous on $(0, 3)$.

(e) If $\frac{1}{x-3}$ is uniformly continuous on $(3,\infty)$, then it is uniformly continuous on $(3,4]$. By Exercise 19.4, $\frac{1}{x-3}$ is bounded on $(3,4]$. However, if we choose $x_n = 3+\frac{1}{n} \in (3,4]$, $n \in \mathbb{N}$, then $\frac{1}{x_n-3} = n \to +\infty$. So $\frac{1}{x-3}$ is not bounded on $(3,4]$. The contradiction shows that $\frac{1}{x-3}$ is not uniformly continuous on $(3,\infty)$.

(f) We calculate the derivative of $\frac{1}{x-3}$ and find

$$\Big|\frac{d}{dx}\frac{1}{x-3}\Big| = \Big|\frac{-1}{(x-3)^2}\Big| = \frac{1}{|x-3|^2} \leq \frac{1}{(4-3)^2} = 1$$

on $[4,\infty)$. So the derivative of $\frac{1}{x-3}$ is bounded on $[4,\infty)$. By Theorem 19.6, $\frac{1}{x-3}$ is uniformly continuous on $[4,\infty)$. □

19.8 (a) Use the Mean Value theorem to prove $|\sin x - \sin y| \leq |x-y|$ for all $x, y \in \mathbb{R}$; see the proof of Theorem 19.6.

(b) Show $\sin x$ is uniformly continuous on $\mathbb{R}$.

Proof. (a) Let $x, y \in \mathbb{R}$. If $x = y$, then $|\sin x - \sin y| = 0 = |x-y|$. Suppose $x \neq y$. Since $|x-y| = |y-x|$ and $|\sin x - \sin y| = |\sin y - \sin x|$, by symmetry we may assume $x < y$. Applying the Mean Value theorem to $\sin x$ on the interval (x,y), we get the existence of $z \in (x,y)$ such that $\frac{\sin x - \sin y}{x-y} = \sin'(z) = \cos(z)$. So $|\frac{\sin x - \sin y}{x-y}| = |\cos(z)| \leq 1$, which implies that $|\sin x - \sin y| \leq |x-y|$.

(b) For $\varepsilon > 0$, let $\delta = \varepsilon > 0$. If $x, y \in \mathbb{R}$ satisfy $|x-y| < \delta$, then by (a), $|\sin x - \sin y| \leq |x-y| < \delta = \varepsilon$. By Definition 19.1, $\sin x$ is uniformly continuous on $\mathbb{R}$. □

20.6 Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to 0} f(x)$ *when they exist* for the function $f(x) = \frac{x^3}{|x|}$. Prove your assertions.

Solution. For $x > 0$, $f(x) = \frac{x^3}{x} = x^2$. If (x_n) is a sequence in $(0,\infty)$ with $x_n \to \infty$, then $f(x_n) = x_n^2 \to \infty$. So $\lim_{x\to\infty} f(x) = \infty$. If (x_n) is a sequence in $(0,\infty)$ and $x_n \to 0$, then $f(x_n) = x_n^2 \to 0$. So $\lim_{x\to 0^+} f(x) = 0$. For $x < 0$, $f(x) = \frac{x^3}{-x} = -x^2$. If (x_n) is a sequence in $(-\infty,0)$ with $x_n \to -\infty$, then $f(x_n) = -x_n^2 \to -\infty$. So $\lim_{x\to-\infty} f(x) = -\infty$. If (x_n) is a sequence in $(-\infty,0)$ and $x_n \to 0$, then $f(x_n) = x_n^2 \to 0$. So $\lim_{x\to 0^-} f(x) = 0$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0$, we get $\lim_{x\to 0} f(x) = 0$. □

20.11 Find the following limits. (a) $\lim_{x\to a} \frac{x^2-a^2}{x-a}$; (b) $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$, $b > 0$; (c) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$. Hint: $x^3 - a^3 = (x-a)(x^2+ax+a^2)$.

Solution. (a) Since $\frac{x^2-a^2}{x-a} = \frac{(x+a)(x-a)}{x-a} = x+a$ for $x \neq a$, we have

$$\lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a}(x+a) = 2a.$$

(b) Since $\frac{\sqrt{x}-\sqrt{b}}{x-b} = \frac{\sqrt{x}-\sqrt{b}}{(\sqrt{x}-\sqrt{b})(\sqrt{x}+\sqrt{b})} = \frac{1}{\sqrt{x}+\sqrt{b}}$ for $x \neq b$, we have

$$\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b} = \lim_{x\to b} \frac{1}{\sqrt{x}+\sqrt{b}} = \frac{1}{2\sqrt{b}}.$$

(c) Since $\frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2+ax+a^2$, we get

$$\lim_{x\to a} \frac{x^3-a^3}{x-a} = \lim_{x\to a}(x^2+ax+a^2) = 3a^2.$$

□

20.16 Suppose that the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exist.

(a) Show if $f_1(x) \le f_2(x)$ for all x in some interval (a,b), then $L_1 \le L_2$. Hint: You may use the results on limits of sequences: If $x_n \to L_1$ and $y_n \to L$, and $x_n \le y_n$ for each n, then $L_1 \le L_2$. See Exercises 8.9 and 9.9.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a,b). Can you conclude that $L_1 < L_2$?

Proof. (a) Let (x_n) be a sequence in (a,b) such that $x_n \to a$. Then $f_1(x_n) \to L_1$, $f_2(x_n) \to L_2$, and $f_1(x_n) \le f_2(x_n)$ for each n. By Exercises 8.9 and 9.9, we get $L_1 \le L_2$.

(b) We can not conclude $L_1 < L_2$ even if $f_1(x) < f_2(x)$ for all x in some interval (a,b). For example, $a=0$, $b=1$, $f_1(x)=0$ and $f_2(x)=x$ on $(0,1)$. Then $f_1(x) < f_2(x)$ on $(0,1)$, but $\lim_{x\to 0^+} f_1(x) = 0 = \lim_{x\to 0^+} f_2(x)$. □

20.17 Show that if $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ and if $f_1(x) \le f_2(x) \le f_3(x)$ for all x in some interval (a,b), then $\lim_{x\to a^+} f_2(x) = L$. Hint: When $L \in \mathbb{R}$, use Exercise 8.5 (Squeeze Lemma for sequences, Theorem 1 in the lecture notes of Sep 16-18). When $L = +\infty$ or $-\infty$, use Exercise 9.9 (the first problem in Homework 4).

Proof. We first prove the following statement. If $x_n \le z_n \le y_n$ for all n, and $\lim x_n = L = \lim y_n$, where L could be a real number or $+\infty$ or $-\infty$, then $\lim z_n = L$. If $L \in \mathbb{R}$, then this follows from the squeeze lemma (Exercise 8.5); if $L = +\infty$, since $x_n \le z_n$ for each n, and $x_n \to +\infty$, by Exercise 9.9 (a), $z_n \to +\infty$; if $L = -\infty$, since $z_n \le y_n$ for each n, and $y_n \to -\infty$, by Exercise 9.9 (b), $z_n \to -\infty$.

Now we return to the proof. Suppose (x_n) is a sequence in (a,b) such that $x_n \to a$. From $\lim_{x\to a^+} f_1(x) = \lim_{x\to a^+} f_3(x) = L$ we know that $\lim f_1(x_n) = \lim f_3(x_n) = L$. From that $f_1(x) \le f_2(x) \le f_3(x)$ on (a,b) we know that $f_1(x_n) \le f_2(x_n) \le f_3(x_n)$ for each n. By the above paragraph, we get $f_2(x_n) \to L$. Since this holds for any sequence (x_n) in (a,b) with $x_n \to a$, we get $\lim_{x\to a^+} f_2(x) = L$. □