MTH 320 Section 004 Midterm 1 Sample

1. Prove the inequality:

$$||x| - |y|| \le |x - y|, \quad \forall x, y \in \mathbb{R}.$$

Proof. Since $x = y + (x - y)$, by triangle inequality, we have $|x| \le |y| + |x - y|$. So

$$|x| - |y| \le |x - y|.$$

Swapping x and y, we get

$$|y| - |x| \le |y - x| = |x - y|,$$

where the equality holds because $y - x = -(x - y)$. Since $||x| - |y||$ equals either $|x| - |y|$ or $-(|x| - |y|) = |y| - |x|$, both of which are $\le |x - y|$, we get the conclusion. □

2. (a) For a nonempty set $S \subseteq \mathbb{R}$, define $\min S$ and $\inf S$.

(b) Let A and B be two nonempty subsets of $\mathbb{R}$. Prove that

$$\inf(A \cup B) = \min\{\inf A, \inf B\}.$$

Here if $\inf A$ or $\inf B$ equals $-\infty$, then we understand $\min\{-\infty, -\infty\}$ and $\min\{-\infty, x\}$ for any $x \in \mathbb{R}$ as $-\infty$.

Solution. (a) $a = \min S$ if $a \in S$ and $a \le s$ for every $s \in S$. If S is bounded below, $\inf S$ is the biggest lower bound of S; if S is not bounded below, $\inf S$ is $-\infty$.

(b) We use a homework problem: For nonempty subsets $S \subset T$ of $\mathbb{R}$, $\inf S \ge \inf T$. Since $A \subset A \cup B$, we get $\inf A \ge \inf(A \cup B)$. Similarly, $\inf B \ge \inf(A \cup B)$. Since $\min\{\inf A, \inf B\}$ equals either $\inf A$ or $\inf B$, we get $\min\{\inf A, \inf B\} \ge \inf(A \cup B)$. We need to prove an inequality in the opposite direction. For any $a \in A$, we have $a \ge \inf A \ge \min\{\inf A, \inf B\}$. For any $b \in B$, we have $b \ge \inf B \ge \min\{\inf A, \inf B\}$. So for any $x \in A \cup B$, we have $x \ge \min\{\inf A, \inf B\}$. This implies that $\inf(A \cup B) \ge \min\{\inf A, \inf B\}$, as desired. □

3. (a) When do we say that a sequence (s_n) converges to s?

(b) Determine the limit of the sequence $s_n = \frac{\cos(n^2)}{2^n}$ and prove your claim.

Solution. (a) We say that (s_n) converges to s if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for any $n > N$, we have $|s_n - s| < \varepsilon$.

(b) We may use squeeze lemma. Since $|\cos(n^2)| \le 1$, we have $-\frac{1}{2^n} \le s_n \le \frac{1}{2^n}$. Since $0 < \frac{1}{2} < 1$, by a theorem in the book, we have $\frac{1}{2^n} = (\frac{1}{2})^n \to 0$. So $-\frac{1}{2^n} \to 0$ as well. Since $-\frac{1}{2^n} \le s_n \le \frac{1}{2^n}$ for every n, we get $s_n \to 0$ by squeeze lemma. □

4. (a) Give the definition of $\limsup s_n$ for a sequence (s_n).

(b) Prove that for any two sequences of nonnegative numbers (s_n) and (t_n),

$$\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n.$$

Here if $\limsup s_n$ or $\limsup t_n = +\infty$, we understand $(+\infty)+(+\infty)$ and $(+\infty)+x$ for any $x \in \mathbb{R}$ as $+\infty$.

Solution. (a) First, we define a sequence $v_N = \sup\{s_n : n > N\}$, $N \in \mathbb{N}$. Then either $v_N = +\infty$ for all N, or (v_N) is a decreasing sequence of real numbers. In the former case, $\limsup s_n$ is defined to be $+\infty$. In the latter case, $\limsup s_n$ is defined to be $\lim_{N\to\infty} v_N$, which always exists because (v_N) is decreasing.

(b) If (s_n) or (t_n) is not bounded above, then at least one of $\limsup s_n$ and $\limsup t_n$ is $+\infty$. In that case, the inequality is trivial. Now we suppose that both (s_n) and (t_n) are bounded above. Then $(s_n + t_n)$ is also bounded above. So for every $N \in \mathbb{N}$,

$$\sup\{s_n + t_n : n > N\}, \quad \sup\{s_n : n > N\}, \quad \sup\{t_n : n > N\}$$

are all finite numbers. Since $s_n, t_n \ge 0$ for all n, these numbers are nonnegative. So the sequences $(\sup\{s_n + t_n : n > N\})_N$, $(\sup\{s_n : n > N\})_N$, $(\sup\{s_n : n > N\})_N$ are all decreasing and bounded below by 0. Thus they respectively converge to $\limsup(s_n+t_n)$, $\limsup s_n$, $\limsup t_n$. Fix $N \in \mathbb{N}$. Then for every $n > N$, $s_n \le \sup\{s_n : n > N\}$ and $t_n \le \sup\{t_n : n > N\}$, and so $s_n + t_n \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. Since this holds for any $n > N$, we get

$$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Since the LHS converges to $\limsup(s_n + t_n)$, and the RHS converges to $\limsup s_n + \limsup t_n$, we get the conclusion. □

5. Let (s_n) be a sequence defined recursively by $s_1 = 10$ and $s_n = \frac{1}{4}(s_{n-1} + 6)$.

(a) Show that (s_n) is decreasing and satisfies $s_n > 2$ for all n.

(b) Does (s_n) converge? If so, what is the limit? Justify your answer carefully.

Proof. (a) We use induction to show that $s_n \ge s_{n+1}$ and $s_n > 2$ for all $n \in \mathbb{N}$. We calculate $s_2 = \frac{1}{4}(s_1 + 6) = \frac{1}{4}(10 + 6) = 4$. When $n = 1$, we have $s_1 = 10 > 4 = s_2$ and $s_1 = 10 > 2$. Suppose the statements hold for n. We now show that they also hold for $n + 1$. In fact, we have

$$s_{n+1} = \frac{1}{4}(s_n + 6) > \frac{1}{4}(2 + 6) = 2,$$

and

$$s_{n+1} - s_{n+2} = \frac{1}{4}(s_n + 6) - \frac{1}{4}(s_{n+1} + 6) = \frac{1}{4}(s_n - s_{n+1}) \ge 0.$$

So by induction $s_n \geq s_{n+1}$ and $s_n > 2$ hold for all $n \in \mathbb{N}$. This means that (s_n) is a decreasing sequence.

(b) Since (s_n) is decreasing and bounded below, it converges. Let $s = \lim s_n$. Then $s = \lim s_{n-1}$. From $s_n = \frac{1}{4}(s_{n-1} + 6)$ and limit theorems, we get $s = \frac{1}{4}(s + 6)$. Solving this equation we get $s = 2$. So the limit of (s_n) is 2. $\square$