MTH 320 Section 004 Midterm 2 Sample

Instructions: You have 60 minutes to complete the exam. There are five problems, worth a total of 20 points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

You may use any result from the book, lecture notes, or homework without proof. You may use the following facts without proof: $\sin x$, $\cos x$, and e^x are all differentiable on $\mathbb{R}$, and the derivatives are $\cos x$, $-\sin x$, and e^x, respectively.

This is a sample exam. The actual exam problems may not be similar to the sample exam problems, and may be harder.

NAME: ____

1. (a) [4pts] Let $f : \mathbb{R} \to \mathbb{R}$. What does it mean to say that f is continuous at x_0?

   (b) [6pts] A set S is said to be *dense* in $\mathbb{R}$ if every open interval contains a point in S. (For example, both the rationals and the irrationals are dense in $\mathbb{R}$.) Suppose S is dense in $\mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$, and $f(s) = g(s)$ for every $s \in S$. Prove that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

2. For each of the following, either give an example of a power series with the given properties, or prove that one cannot exist. The center does not have to be 0.

   (a) [3pts.] A power series with interval of convergence $(0, 2]$.

   (b) [4pts.] A power series which converges uniformly on its interval of convergence.

   (c) [3pts.] A power series with interval of convergence $(-2, 2)$.

3. (a) [4pts] Let $f : \mathbb{R} \to \mathbb{R}$. What does it mean to say that f is differentiable at x_0?

   (b) [6pts] Prove that $f(x) = \cos(\sin(x^3) + e^{\frac{1}{x^2}})$ is differentiable on $\mathbb{R} \setminus \{0\}$, and compute $f'(x)$. Carefully justify each step.

4. (a) [4pts] What is the Weierstrass M-test?

(b) [6pts] Suppose that a power series $\sum a_n x^n$ has radius of convergence $R > 0$. Let $0 < R_0 < R$. Prove that the series $\sum a_n x^n \cos(x^2)$ converges uniformly on $[-R_0, R_0]$ to a continuous function.

5. Let (a_n) be a sequence of positive numbers such that $\lim a_n = 0$. (a) [5pts.] Give an example to show that $\sum a_n$ need not converge. (b) [5pts.] Prove that there exists a subsequence (a_{n_k}) of (a_n) such that $\sum a_{n_k}$ converges.

This page is for scratch work. Feel free to tear it off. Do not write anything you want graded on this page unless you indicate very clearly that this is the case on the page of the corresponding problem.