Quiz 1 Solutions

1. (i) State the Rational Zeros Theorem.
   (ii) Find all rational solutions of the polynomial equation

$$x^3 + 2x^2 - 11x - 12 = 0.$$

Solution. (i) It is Theorem 2.2 on Page 9 of the book.

(ii) By Corollary 2.3, if a rational number r is a solution of the equation, then r is an integer and divides the constant term -12 in the equation. So the possible value of r is

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 4, \quad \pm 6, \quad \pm 12.$$

Not everyone of the above is an actual solution. You have to plug them into the equation one by one to check them. This takes some work. You will find that -1, 3, and -4 are solutions. Since this is a polynomial of order 3, it has at most three solution. Then you do not need to check other solutions. □

2. (i) Let S be a nonempty subset of $\mathbb{R}$. State the definition of $\inf S$. Please cover all cases.
   (ii) Prove that

$$\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0.$$

Solution. (i) If S is bounded below, then $\inf S$ is the greatest lower bound of S. If S is not bounded below, then $\inf S = -\infty$.

(ii) Let $S = \{\frac{1}{n} : n \in \mathbb{N}\} = 0$. We want to show $\inf S = 0$. Using the definition in (i), we want to show that 0 is a lower bound of S, and is the biggest among all lower bounds of S. Since $\frac{1}{n} > 0$ for every $n \in \mathbb{N}$, we see that 0 is a lower bound of S. Suppose 0 is not the biggest lower bound of S. Then there is another lower bound of S, say a, which satisfies $a > 0$. We derive a contradiction from this. Since a is a lower bound of S, we have $\frac{1}{n} \geq a$ for every $n \in \mathbb{N}$. However, since $a > 0$, by Archimedean property, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < a$. This contradicts that $\frac{1}{n} \geq a$ for every $n \in \mathbb{N}$. So such a does not exist, and 0 is the biggest lower bound, i.e., $0 = \inf S$. □