Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 13.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. (14 points) Find the length L of the curve given by $x(t) = \ln(\sec t)$ from $t = 0$ to $t = \frac{\pi}{4}$.

Solution:

$$x'(t) = \left(\frac{1}{\sec(t)}\right)\sec(t)\tan(t) = \tan(t)$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2(t)}\; dt$$

$$= \int_0^{\pi/4} \sec(t)\; dt$$

(using that $\sec^2(t) = 1 + \tan^2(t)$)

$$= \ln|\sec(t)+\tan(t)|\;\Big|_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| + \ln|1+0|$$

$$= \ln\left(\sqrt{2}+1\right)$$

2. Determine whether the **sequences** below converge or diverge. If a sequence converges, find its limit. If you use L'Hopital's Rule, explicitly state your reasoning.

   (a) (7 points) $a_n = n\sin(1/n)$

   Solution:

$$\lim_{n\to\infty} n\sin(1/n)$$

$$= \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \quad \frac{0}{0}\text{case}$$

$$= \lim_{n\to\infty} \frac{-\cos(1/n)n^{-2}}{-n^{-2}}$$

$$= 1$$

   (b) (7 points) $b_n = \ln(2n+3) - \ln(n)$

   Solution:

$$\lim_{n\to\infty} \ln(2n+3) - \ln(n)$$

$$= \lim_{n\to\infty} \ln\left(2 + \frac{3}{n}\right)$$

$$= \ln\left(\lim_{n\to\infty} 2 + \frac{3}{n}\right)$$

$$= \ln(2)$$

3. Determine if the following series converge. Make sure to justify your answer, show your work, and explicitly state which test(s) you use.

(a) (7 points) $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$

Solution:

$$\sum_{k=1}^{\infty} \left| \frac{\sin k}{k^3} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ converges by } p - \text{series test}$$

$=$ The series converges (absolutely), and so converges

(b) (7 points) $\sum_{k=3}^{\infty} \frac{1}{k\sqrt{\ln k}}$

Solution:

$$\frac{1}{k\sqrt{\ln k}} \text{ decreases with } k$$

$$\text{Use integral test: } \int_3^{\infty} \frac{dx}{x\sqrt{\ln x}}$$

$$= \lim_{a \to \infty} \int_{\ln 3}^{a} \frac{du}{\sqrt{u}}$$

$$= \lim_{a \to \infty} 2a^{1/2} - 2\sqrt{\ln 3} = \infty$$

$=$ The integral diverges, so the series does too

4. Answer the following questions about power series. Justify your answers and show your work!

(a) (7 points) Does $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n)!}$ converge at $x = 100$?

Solution:

Use the ratio test on $\displaystyle\sum_{n=0}^{\infty} \frac{99^n}{(2n)!}$

$$= \lim_{n\to\infty} \left| \frac{99^{n+1}}{(2n+2)!} \frac{(2n)!}{99^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{99}{(2n+2)(2n+1)} \right|$$

$$= 0 < 1, \text{ so converges}$$

– Also full points if they show radius of convergence = ∞.

(b) (7 points) Find the open interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}-1\right)^n}{n^{10}}$.

Solution:

Use the ratio test...

$$\text{Need } \lim_{n\to\infty} \left| \frac{\left(\frac{x}{2}-1\right)^{(n+1)}}{(n+1)^{10}} \frac{n^{10}}{\left(\frac{x}{2}-1\right)^n} \right| < 1$$

$$\iff \lim_{n\to\infty} \left| \left(\frac{x}{2}-1\right) \frac{n^{10}}{(n+1)^{10}} \right| < 1$$

$$\iff \left| \left(\frac{x}{2}-1\right) \right| < 1$$

$$\iff x \in (0, 4)$$

5. (14 points) Find the 2^{nd} degree Taylor polynomial of $f(x) = \ln(\sin x)$ centered at $x = \frac{\pi}{4}$.

Solution:

$$f(\pi/4) = \ln(\sin \pi/4) = \ln(1/\sqrt{2})$$

$$f'(x) = \frac{1}{\sin(x)} \cos(x) = \frac{\cos(x)}{\sin(x)}$$

$$f'(\pi/4) = \frac{\cos(\pi/4)}{\sin(\pi/4)} = 1$$

$$f''(x) = -1 - \frac{\cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2 x}$$

$$f''(\pi/4) = -1 - \frac{\cos^2(\pi/4)}{\sin^2(\pi/4)} = -\frac{1}{\sin^2(\pi/4)} = -2$$

Approximation is $\ln(1/\sqrt{2}) + (x - \pi/4) - (x - \pi/4)^2$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (4 points) Which of the following series converge?

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+n}$

(2) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

(3) $\displaystyle\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \dots$

A. (1) only

B. (2) only

C. (3) only

D. (1) and (2) only

E. (2) and (3) only

F. They all converge

7. (4 points) Evaluate the limit: $\displaystyle\lim_{x\to 0^+} \frac{e^{2x}-1-2x}{\ln(1+x^2)}$

A. 4

B. 3

C. 2

D. 1

E. 0

8. (4 points) Which of the following statements is true?

(1) If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

(2) If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

(3) If $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

A. (1) only

B. (2) only

C. (3) only

D. (1) and (2) only

E. (2) and (3) only

F. They are all true

9. (4 points) Find the sum of the series $\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^{2n}}$

A. 4

B. 3

C. 4/3

D. 1

E. 0

10. (4 points) Which of the following statements are true?

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ converges absolutely.

(2) $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(3) $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ converges by the ratio test.

A. Only (1) is true.

B. Only (2) is true.

C. Only (3) is true.

D. Only (1) and (2) are true.

E. Only (2) and (3) are true.

F. They are all true

11. (4 points) The power series $\sum_{n=1}^{\infty} (-2)^n x^n$ converges for $|x| < \frac{1}{2}$ to

A. $\frac{-2x}{1+2x}$

B. $\frac{2x}{1+2x}$

C. $\frac{1}{1+2x}$

D. $\frac{1}{1-2x}$

E. $\frac{-1}{1-2x}$

12. (4 points) What is the approximation of the value of $\sin 1$ obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

A. $1 - \frac{1}{2} + \frac{1}{24}$

B. $1 - \frac{1}{2} + \frac{1}{4}$

C. $1 - \frac{1}{3} + \frac{1}{5}$

D. $1 - \frac{1}{4} + \frac{1}{8}$

E. $1 - \frac{1}{6} + \frac{1}{120}$

13. (4 points) If $|f^{(4)}| < 1$ on the interval $[-5, 5]$, the difference between $f(4.5)$ and it's third degree Maclaurin polynomial evaluated at $x = 4.5$ is always less than which of the following:

A. $\frac{(4.5)^3}{3!}$

B. $\frac{1}{3!}$

C. $\frac{(4.5)^4}{4!}$

D. $\frac{1}{4!}$

E. $\frac{(4.5)^5}{5!}$

14. (4 points) Evaluate the indefinite integral $\int e^{x^2} dx$ as a power series.

A. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} + C$

B. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} + C$

C. $\sum_{n=0}^{\infty} \frac{(2n-1)x^{2n-1}}{n!} + C$

D. $\sum_{n=0}^{\infty} \frac{x^{2n-1}}{(2n-1)n!} + C$

E. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!} + C$

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	14	
7	12	
9	12	
10	12	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET PAGE 1

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\int \frac{1}{x}\ dx = \ln|x| + C$
- $\int \tan x\ dx = \ln|\sec x| + C$
- $\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\int a^x\ dx = \frac{a^x}{\ln a} + C$ for $a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

- Arc Length Formula:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

Derivatives

- $\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad \frac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \text{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

FORMULA SHEET PAGE 2

Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.
- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then
  - (i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  - (ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  - (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  - (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.
- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies
  - (i) $0 < b_{n+1} \leq b_n$ for all n
  - (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.
- **The Ratio Test**
  - (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.
- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad R = 1$