Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (7 points) The region in the first quadrant bounded by the $y = x^3$, $y = \sqrt{x}$, is rotated about the $y-$axis. Compute the volume of the resulting solid of revolution.

Solution: Convert first to $x = \sqrt[3]{y}$ and $x = y^2$ then we can see that these curves intersect at $(0, 0)$ and $(1, 1)$ so the volume is given by

$$\begin{aligned} V &= \pi \int_0^1 (\sqrt[3]{y})^2 - (y^2)^2 \; dy \\ &= \pi \int_0^1 y^{2/3} - y^4 \; dy \\ &= \pi \left[\frac{3}{5} y^{5/3} - \frac{1}{5} y^5 \right] = \pi \frac{35}{5} - \pi \frac{1}{5} = \boxed{\frac{2\pi}{5}} \end{aligned}$$

2. (7 points) Find the function $y = y(x)$ which solves the initial value problem

$$\frac{dy}{dx} = \frac{y}{2 + x^2} \quad \text{and} \quad y(0) = 1.$$

Solution: First we can rearrange to get

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{2+x^2} && \text{(now integrate)} \\ \int \frac{dy}{y} &= \int \frac{dx}{2+x^2} && \text{(find anti-derivatives)} \\ \ln|y| &= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C && \text{(plug in } x = 0 \text{ and } y = 1) \\ 0 &= 0 + C && \text{(plug } C = 0 \text{ back into our equation)} \\ \ln|y| &= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) && \text{(solve for } y) \\ y &= e^{\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)} \end{aligned}$$

3. (7 points) A swimming pool built in the shape of a rectangular prism 3 m deep, 5 m wide and 10 m long is filled 1 m below the top. How much work is required to pump all the water into a drain at the top edge of the pool ? (water weighs 9800 N/m^3).

Solution: Use $\sigma = 9800$, $A(y) = 5(10) = 50$ and $d(y) = 3 - y$ to give us

$$\begin{aligned} W &= \int_0^2 9800 \cdot 50 \cdot (3-y)\, dy \\ &= 490000 \left[3y - y^2/2\right]_0^2 \\ &= 490000\,[6-2] \\ &= 490000 \cdot 4 = \boxed{1960000 \text{ N m}} \end{aligned}$$

4. (7 points) Two positively charged particles repel each other with a force of $F(x) = \dfrac{1}{x^2}$ where x is the distance between the two. One particle is held fixed while the other is pushed from a point 1 m away from the first particle to a point 0.1 m away from the first particle. How much work is done ?

Solution:

$$\begin{aligned} W &= \int_1^{1/10} \frac{1}{x^2}\, dx \\ &= \left[-\frac{1}{x}\right]_1^{1/10} \\ &= [-10+1] = \boxed{-9 \text{ units of Work}} \qquad \text{(negative since done in the direction opposite the force)} \end{aligned}$$

5. (7 points) Find the derivative of $f(x) = (x+1)^{\ln(\mathrm{x})}$.

Solution: Consider $y = (x+1)^{\ln(\mathrm{x})}$ then

$$\begin{aligned}
\ln y &= \ln(x+1)^{\ln(\mathrm{x})} \\
\ln y &= \ln x \cdot \ln(x+1) \\
\frac{1}{y} \cdot y' &= \frac{1}{x} \cdot \ln(x+1) + \ln x \cdot \frac{1}{x+1} \\
y' &= (x+1)^{\ln(\mathrm{x})} \left(\frac{1}{x} \cdot \ln(x+1) + \ln x \cdot \frac{1}{x+1} \right)
\end{aligned}$$

6. (7 points) Evaluate $\displaystyle\int \frac{3x^2+2}{x^3+2x-5}\, dx$.

Solution: Let $u = x^3 + 2x - 5$ then $du = 3x^2 + 2x\ dx$ then

$$\begin{aligned}
\int \frac{3x^2+2}{x^3+2x-5}\, dx &= \int \frac{1}{u}\, du \\
&= \ln|u| + C \\
&= \ln|x^3 + 2x - 5| + C
\end{aligned}$$

7. (8 points) After an accident at a nuclear facility a sample of sea water was tested for radioactive contamination, and it showed an elevated level of 27 Bq/m^3 (Becquerel per cubic meter, a measure of radiation) primarily due to Cesium-137 which has a half life of 30 years. How many years does it take for the radiation to return to its natural level of 1 Bq/m^3 ? (Use the approximate values 10/3 and 7/10 for $\ln 27$ and $\ln 2$ respectively)

Solution: We can start with the equation $A(t) = 27e^{kt}$ then since the half life is 30 years we get

$$\frac{27}{2} = 27e^{k\cdot 30}$$
$$\frac{1}{2} = e^{k\cdot 30}$$
$$\ln\tfrac{1}{2} = k\cdot 30$$
$$\ln\tfrac{1}{2} = k\cdot 30 \longrightarrow \frac{\ln\frac{1}{2}}{30} = k$$

From here we can solve

$$1 = 27e^{\frac{\ln\frac{1}{2}}{30}\cdot t}$$
$$\frac{1}{27} = e^{\frac{\ln\frac{1}{2}}{30}\cdot t}$$
$$\ln\tfrac{1}{27} = \frac{\ln\frac{1}{2}}{30}\cdot t$$
$$\frac{30\ln\frac{1}{27}}{\ln\frac{1}{2}} = t$$

using the approximate values specified we get

$$t = \frac{30\ln\frac{1}{27}}{\ln\frac{1}{2}} = \frac{30\ln 27}{\ln 2} = 30\cdot\frac{10/3}{7/10} = \boxed{\frac{1000}{7}\text{ years}}$$

8. (6 points) Evaluate $\lim_{x\to 0}\frac{\ln(\sec x)}{x^2}$. (*If you use l'Hospital's Rule, explicitly state your reasoning.*)

Solution: Since $\lim_{x\to 0}\frac{\ln(\sec x)}{x^2} \to \frac{0}{0}$ so l'Hopital's rule applies.

$$\lim_{x\to 0}\frac{\ln(\sec x)}{x^2} \overset{LH}{=} \lim_{x\to 0}\frac{\frac{1}{\sec x}\cdot\sec x\tan x}{2x}$$
$$= \lim_{x\to 0}\frac{\tan x}{2x} \qquad \text{(which is also a 0/0 indeterminant)}$$
$$\overset{LH}{=} \lim_{x\to 0}\frac{\sec^2 x}{2} = \boxed{\frac{1}{2}}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

9. (4 points) What is the value of x if $\displaystyle\int_1^{x^2} \frac{dt}{t} = -1$?

A. e

B. $\dfrac{1}{e}$

C. $\dfrac{1}{\sqrt{e}}$

D. $\sqrt{e}$

E. 0

10. (4 points) What is the value of $\cos(\tan^{-1}(3))$?

A. $\dfrac{1}{\sqrt{10}}$

B. $\dfrac{3}{\sqrt{10}}$

C. $\dfrac{1}{\sqrt{3}}$

D. $\dfrac{10}{\sqrt{3}}$

E. $\dfrac{2}{\sqrt{10}}$

11. (4 points) What is the derivative of $y = \sin^{-1}(\ln x)$?

A. $\dfrac{1}{x\sqrt{1-x^2}}$

B. $\dfrac{1}{x\sqrt{1-\ln^2 x}}$

C. $\dfrac{1}{\ln x\sqrt{1-x^2}}$

D. $\dfrac{1}{x(1+x\ln^2 x)}$

E. $\dfrac{1}{\ln x(1+x^2)}$

12. (4 points) Evaluate $\int_0^{\ln 2} \sinh 2x \; dx$.

A. $\frac{17}{16}$

B. $\frac{1}{2}$

C. $\frac{15}{16}$

D. $\frac{9}{16}$

E. $\frac{1}{16}$

13. (4 points) Evaluate $\int \sin^4 x \cos^3 x \; dx$.

A. $\frac{1}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

B. $\frac{1}{7}\sin^7 x - \frac{1}{5}\sin^5 x + C$

C. 0

D. $-\frac{1}{20}\cos^5 x \sin^4 x + C$

E. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

14. (4 points) Let $f(x) = x^3 + 2\sqrt{x} + 2$. What is $(f^{-1})'(5)$?

A. 7/2

B. $75 + \frac{1}{\sqrt{5}}$

C. 1/5

D. 1/4

E. 1/2

15. (4 points) Determine the values for A and B, such that $y = \dfrac{1}{x(10+x)} = \dfrac{A}{x} + \dfrac{B}{10+x}$.

A. $A = \dfrac{1}{10}, B = 10$

B. $A = 10, B = \dfrac{1}{10}$

C. $A = 1, B = 1$

D. $A = \dfrac{1}{10}, B = \dfrac{1}{10}$

E. $A = \dfrac{1}{10}, B = -\dfrac{1}{10}$

16. (4 points) Which of the following integrals is improper?

A. $\displaystyle\int_0^1 \frac{dt}{\ln(2+x)}$

B. $\displaystyle\int_0^1 \frac{1}{x^2-4}\, dx$

C. $\displaystyle\int_0^1 \frac{1}{t} dx$

D. $\displaystyle\int_0^1 \frac{1}{x^2-1} dx$

E. $\displaystyle\int_{-1}^1 \frac{1}{x^2-4} dx$

17. (4 points) Evaluate $\displaystyle\int_0^{\pi/2} x \sin x \; dx$.

A. 1

B. $\dfrac{\pi}{2}$

C. -1

D. $-\dfrac{\pi}{2}$

E. 0

More Challenging Questions

18. (4 points) **Multiple Choice** Consider the following two unrelated statements:

   1. If $f(t) \leq g(t)$ and $\int_1^\infty f(t)\,dt$ diverges, then $\int_1^\infty g(t)\,dt$ diverges.
   2. $\left(\dfrac{\log_3 x}{\log_7 x}\right)' = 0$

   A. both statements are true

   B. 1. is true and 2. is false

   C. 1. is false and 2. is true

   D. both statements are false

Solution: 1. is false because it does not exclude negatives. A counterexample would be $f(x) = -x$ and $g(x) = 0$.
2. is true since $\dfrac{\log_3 x}{\log_7 x} = \dfrac{\ln 7}{\ln 3}$

19. (10 points) Compute $\int \sqrt{x}\tan^{-1}\sqrt{x}\,dx$. Show all your work!

Solution: Consider integration by parts with $u = \tan^{-1}\sqrt{x}$ and $dv = \sqrt{x}$. Then $du = \dfrac{1}{1+x}\cdot\dfrac{dx}{2\sqrt{x}}$ and $v = \dfrac{2}{3}x^{3/2}$ giving us

$$\begin{aligned}
\int \sqrt{x}\tan^{-1}\sqrt{x}\,dx &= \frac{2}{3}x^{3/2}\cdot\tan^{-1}\sqrt{x} - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{1+x}\cdot\frac{dx}{2\sqrt{x}} \\
&= \frac{2}{3}x^{3/2}\cdot\tan^{-1}\sqrt{x} - \frac{1}{3}\int \frac{x}{1+x}dx \\
&= \frac{2}{3}x^{3/2}\cdot\tan^{-1}\sqrt{x} - \frac{1}{3}\int 1 - \frac{1}{1+x}dx \\
&= \frac{2}{3}x^{3/2}\cdot\tan^{-1}\sqrt{x} - \frac{1}{3}\left[x - \ln|1+x|\right] + C
\end{aligned}$$