Name: ______________________________

Section: ____________ **Recitation Instructor**: ______________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Determine if the following series converge or diverge. **If the series converges, also compute the sum**. You must show all of your work and support your conclusions.

(a) (7 points) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n}$

Solution:
This series looks like a geometric series with $r = 3/4 < 1$, so it should converge. We can compute its value as follows:

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^n = \frac{3}{1-\frac{3}{4}} = \boxed{12}.$$

(b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

Solution:
Consider the n^{th}**-term test**:
We compute the limit of the terms

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0.$$

Because the limit of the terms is nonzero the series $\boxed{\text{diverges.}}$

2. Determine if the following series converge or diverge. You must show all of your work and justify your use of any series convergence tests.

   (a) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

Solution:

Use the **ratio test**:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^5}{5^{n+1}} \cdot \frac{5^n}{n^5} \\
&= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \frac{1}{5} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^5 \cdot \frac{1}{5} \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^5 \cdot \frac{1}{5} \\
&= \frac{1}{5} < 1.
\end{aligned}$$

Thus, the series $\boxed{\text{converges.}}$

   (b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Solution:

Consider the **ratio test**:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{1}{2^{n+1} - 1} \cdot 2^n - 1 \\
&= \lim_{n\to\infty} \frac{2^n - 1}{2^{n+1} - 1} \\
&= \lim_{n\to\infty} \frac{1 - \frac{1}{2^n}}{2 - \frac{1}{2^n}} \\
&= \frac{1}{2} < 1
\end{aligned}$$

Thus, the series $\boxed{\text{converges.}}$

3. For each of the following functions, find its 3rd degree Taylor polynomial centered at the given a.

(a) (7 points) $f(x) = e^{-5x}$, centered at $a = 0$.

Solution:

From, e.g., the formula sheet we can see that

$$e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5x)^n}{n!}.$$

For the third degree Taylor polynomial centered at zero we need to write down the first 4 terms

$$\boxed{T_3(x) = 1 + -5x + \frac{25x^2}{2!} + \frac{-125x^3}{3!} \quad = \quad \frac{-125x^3}{6}x^3 + \frac{25}{2}x^2 - 5x + 1.}$$

(b) (7 points) $g(x) = \ln(x)$, centered at $a = 1$.

Solution:

From, e.g., the formula sheet we can see that

$$\ln(x) = \ln(1 + (x-1)) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}.$$

For the third degree Taylor polynomial centered at zero we need to write down the first 3 terms

$$\begin{aligned} \boxed{T_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}} \quad &= \quad (x-1) - \frac{x^2 - 2x + 1}{2} + \frac{x^3 - 3x^2 + 3x - 1}{3} \\ &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x - \frac{10}{3}. \end{aligned}$$

4. (7 points) Find the Maclaurin series (Taylor series centered at $a = 0$) representation of $f(x) = \dfrac{x}{1+9x^2}$. Express your answer in sigma notation.

Solution:
$\dfrac{x}{1+9x^2} = \dfrac{a}{1-r}$ with $a = x$ and $r = -9x^2$.
So $f(x)$ can be written as the geometric series.

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} x(-9x^2)^n = \boxed{\sum_{n=0}^{\infty} (-9)^n x^{2n+1}}.$$

5. (7 points) Find the interval of convergence for the power series $\displaystyle\sum_{n=3}^{\infty} \frac{(3x-5)^n}{n-2}$. (Leave your answer as an open interval; you do not have to test the end points for convergence.)

Solution:
We should apply the **ratio test**:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(3x-5)^{n+1}}{n-1} \cdot \frac{n-2}{(3x-5)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{n-2}{n-1}(3x-5)\right| \\
&= \lim_{n\to\infty} \left|\frac{1-\frac{2}{n}}{1-\frac{1}{n}}(3x-5)\right| \\
&= |3x-5|.
\end{aligned}$$

For convergence we need $|3x-5| < 1 \implies -1 < 3x-5 < 1 \implies 4 < 3x < 6 \implies \frac{4}{3} < x < 2$.
So, our open interval is $\boxed{\left(\frac{4}{3}, 2\right)}$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (4 points) Find the limit of the **sequence** a_n where the n$^{\text{th}}$ term is given by $a_n = \dfrac{3\sin 4n}{4n}$.

   A. 0

   B. $\frac{3}{4}$

   C. 3

   D. 4

   E. The sequence diverges

7. (4 points) Which statement about the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n}}{n^3+1}$ is true?

   A. It diverges by using the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   B. It converges by using the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

   C. It diverges by using the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   D. It converges by using the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   E. It diverges by the ratio test.

8. (4 points) Which statement is true about the series $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n(\ln(n)+1)^3}$?

   A. The **integral test** shows that the series converges.

   B. The **integral test** shows that the series diverges.

   C. The **integral test** hypotheses are not met by this series, so it cannot be applied.

   D. The **integral test** hypotheses are met by this series, however the test is inconclusive.

   E. None of the above are true.

9. (4 points) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(4x+7)^{3n+1}}{n!}$.

   A. 1/4

   B. 1/3

   C. 3

   D. 4

   E. $+\infty$

10. (4 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1) \sum_{n=1}^{\infty} \frac{\cos n}{n^2+1} \quad \text{and} \quad (2) \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

   A. (1) is absolutely convergent; (2) is divergent.

   B. (1) is conditionally convergent; (2) is divergent.

   C. (1) is absolutely convergent; (2) is conditionally convergent.

   D. (1) is divergent; (2) is conditionally convergent.

   E. Both (1) and (2) are conditionally convergent.

11. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$?

   A. By the ratio test, the series converges.

   B. By the ratio test, the series diverges.

   C. The ratio test is inconclusive for this series, but the series converges by another test.

   D. The ratio test is inconclusive for this series, but the series diverges by another test.

   E. None of the above are true.

12. (4 points) Which of the following series converge?

(1) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ (2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$ (3) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

A. All three.

B. Only (1) and (2).

C. Only (3).

D. Only (2) and (3).

E. None.

13. (4 points) The Taylor series of the function $f(x)$, centered at $a = 2$, is given by $\sum_{n=0}^{\infty} \frac{n^2-n+1}{n!}(x-2)^n$. What is the value of the third derivative $f'''(2)$?

A. 1

B. 3/2

C. 3

D. 7/6

E. 7

14. (4 points) The first 4 nonzero terms of the Taylor series, centered at $a = 0$, for $f(x) = 10x\cos x - 2 + 5x^3$ are

A. $-2 + 10x - \frac{10x^2}{2!} + \frac{10x^4}{4!}$

B. $10x - \frac{10x^3}{2!} + \frac{10x^5}{4!} - \frac{10x^7}{6!}$

C. $-2 + 10x - \frac{10x^3}{2!} + \frac{10x^5}{4!}$

D. $-2 + 10x + \frac{10x^5}{4!} - \frac{10x^7}{6!}$

E. $-2 + 10x^3 - \frac{10x^5}{4!} + \frac{10x^7}{6!}$

More Challenging Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

15. (a) (4 points) Give an example of a series that converges but does not converge absolutely; and give a brief justification.

Solution:
The simplest solution to this problem is to use an alternating p-series with any $\boxed{0 < p \leq 1}$:
In that case,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges by the p-series test, while

$$\boxed{\sum_{n=1} \frac{(-1)^n}{n^p}}$$

converges by the alternating series test.

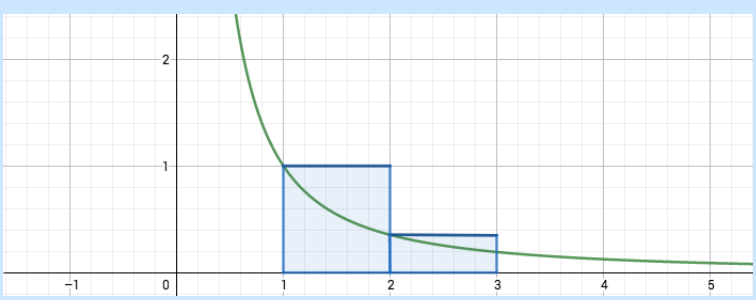
(b) (4 points) Give an example of a convergent series where the ratio test is inconclusive; and give a brief justification.

Solution: Again, the simplest solution to this problem is to use a p-series with any $\boxed{p > 1}$:
In that case, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p}}$ converges by the p-series test, while the ration test is inconclusive since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n^p}{(n+1)^p} \right| = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^p = 1.$$

16. (6 points) Which is larger: $\displaystyle\int_1^{\infty} \frac{1}{x^{3/2}}\, dx$ or $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$? Explain.

Solution: The $\boxed{\text{series is bigger}}$ because it serves as an upper Riemann sum approximation to the integral, and $\frac{1}{x^{3/2}}$ decreases monotonically. Something like following picture can be used as supporting evidence for this claim:



It should be made clear that the blue boxes above have the same area as the first two terms of the series, etc...

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET PAGE 1

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$
- $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\ dx = \frac{a^x}{\ln a} + C \qquad \text{for } a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

- Arc Length Formula:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

Derivatives

- $\dfrac{d}{dx}(\sinh x) = \cosh x \qquad \dfrac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

FORMULA SHEET PAGE 2

Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then
  - (i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  - (ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
  - (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  - (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies
  - (i) $0 < b_{n+1} \leq b_n$ for all n
  - (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.

- **The Ratio Test**
  - (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n$

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad R = 1$