Name: ______________________________

Section: ______________ **Recitation Instructor**: ______________________________

INSTRUCTIONS

- Fill in your name, etc. on this first page.
- Without fully opening the exam, check that you have pages 1 through 12.
- **Show all your work** on the standard response questions. Write your answers clearly! Include enough steps for the grader to be able to follow your work. Don't skip limits or equal signs, etc. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand.
- You will be given exactly 90 minutes for this exam.
- Remove and utilize the formula sheet provided to you at the end of this exam.

ACADEMIC HONESTY

- Do not open the exam booklet until you are instructed to do so.
- Do not seek or obtain any kind of help from anyone to answer questions on this exam. If you have questions, consult only the proctor(s).
- Books, notes, calculators, phones, or any other electronic devices are not allowed on the exam. Students should store them in their backpacks.
- No scratch paper is permitted. If you need more room use the back of a page. You must indicate if you desire work on the back of a page to be graded.
- Anyone who violates these instructions will have committed an act of academic dishonesty. Penalties for academic dishonesty can be very severe. All cases of academic dishonesty will be reported immediately to the Dean of Undergraduate Studies and added to the student's academic record.

I have read and understand the above instructions and statements regarding academic honesty: ______________________________

SIGNATURE

Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. (7 points) The region in the first quadrant bounded by the $y = \sin x$, $y = \cos x$, and the $y-$axis is rotated about the $x-$axis. Compute the volume of the resulting solid of revolution. *Hint:* $\cos^2 x - \sin^2 x = \cos 2x$.

Solution: First find the intercept between the two curves: $\sin x = \cos x$ when $x = \dfrac{\pi}{4}$.
Then:

$$\begin{aligned} V &= \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x)\ dx \\ &= \int_0^{\pi/4} \pi \cos 2x\ dx \\ &= \frac{\pi}{2} \sin 2x \Big|_0^{\pi/4} \\ &= \frac{\pi}{2} \end{aligned}$$

2. (7 points) Let $f(x) = x^3 + x - 2$. Knowing that $f(1) = 0$, what is $(f^{-1})'(0)$?

Solution:

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 1} = \frac{1}{4}$$

3. A force of 0.2 N is required to hold a spring 0.3 m beyond its natural length.

(a) (7 points) Using Hooke's Law, determine the value (with units) of the constant k.

Solution: $F = kx$, therefore $0.2 = 0.3k$, which gives $k = \dfrac{2}{3}$ N/m.

(b) (7 points) How much work is done in stretching the spring from its natural length to 0.5 m beyond its natural length? Specify units.

Solution:

$$\begin{aligned} W &= \int_0^{0.5} kx\ dx \\ &= \frac{2}{3}\int_0^{0.5} x\ dx \\ &= \frac{2}{3}\cdot\frac{1}{2}x^2\Big|_0^{0.5} = \frac{1}{12}\text{ J} \end{aligned}$$

4. (7 points) Find a formula for the instantaneous rate of change of $y = (x+1)^x$.

Solution:

$$\begin{aligned} \ln(y) &= x\ln(x+1) \\ \frac{y'}{y} &= \ln(x+1) + \frac{x}{x+1} && \text{(differentiating both sides)} \\ y' &= \boxed{(x+1)^x\left(\ln(x+1) + \frac{x}{x+1}\right)} \end{aligned}$$

5. (7 points) Evaluate $\displaystyle\int e^{\sin 2t}\cos 2t\ dt$.

Solution: Using the $u-$substitution $u = \sin 2t$, $du = 2\cos 2t dt$, the given integral becomes:

$$\begin{aligned} \frac{1}{2}\int e^u\ du &= \frac{1}{2}e^u + C \\ &= \boxed{\frac{1}{2}e^{\sin 2t} + C} \end{aligned}$$

6. (8 points) Use the method of carbon dating to determine the age of a fossil in which the radioactive carbon has decayed to 15% of its original amount. Assume the half-life of the radioactive carbon is 6000 years.

Solution:

$$\begin{aligned} m(6000) = \frac{1}{2}m_0 &= m_0 e^{-6000k} \\ \frac{1}{2} &= e^{-6000k} \\ \ln\frac{1}{2} &= -6000k \\ k &= \frac{\ln 2}{6000} \end{aligned}$$

Therefore $m(t) = m_0 e^{-\frac{\ln 2}{6000}t}$. So then

$$\begin{aligned} m(t) = 0.15m_0 &= m_0 e^{-\frac{\ln 2}{6000}t} \\ 0.15 &= e^{-\frac{\ln 2}{6000}t} \\ t &= -\frac{\ln 0.15}{\frac{\ln 2}{6000}} \end{aligned}$$

7. (6 points) Evaluate $\lim_{x\to 0} \frac{\ln(\cos x)}{x}$. (*If you use l'Hospital's rule, explicitly state your reasoning.*)

Solution: Since as $x \to 0$ we have $\cos(x) \to 1$ and so $\ln(\cos(x)) \to 0$. So this limit is indeterminate of the form 0/0.

$$\begin{aligned} \lim_{x\to 0} \frac{\ln(\cos x)}{x} &= \lim_{x\to 0} \frac{\left(\frac{-\sin x}{\cos x}\right)}{1} && \text{(l'Hospital)} \\ &= \lim_{x\to 0} \frac{-\tan x}{1} = \boxed{0} \end{aligned}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

8. (4 points) What is the value of x if $\int_1^x \frac{dt}{t} = -1$?

A. e

B. $\frac{1}{e}$

C. $\frac{1}{e} + C$

D. 1

E. 0

9. (4 points) What is the value of $\sin(\sec^{-1}(3))$?

A. $\frac{8}{9}$

B. $\frac{2}{3}$

C. $\frac{2\sqrt{2}}{3}$

D. $\frac{1}{2}$

E. $\frac{1}{3}$

10. (4 points) What is the derivative of $y = \tan^{-1}\sqrt{x}$?

A. $\frac{1}{1+x^2}\frac{1}{2\sqrt{x}}$

B. $-\frac{1}{1+x}\frac{1}{2\sqrt{x}}$

C. $\frac{1}{\sqrt{1+x}}\frac{1}{2\sqrt{x}}$

D. $\frac{1}{1+x}\frac{1}{2\sqrt{x}}$

E. $\frac{1}{\sqrt{1-x}}\frac{1}{2\sqrt{x}}$

11. (4 points) Evaluate $\int_0^{\ln 3} \cosh 2x \, dx$.

A. 2.

B. 0.

C. $\frac{80}{9}$.

D. $\frac{40}{9}$.

E. $\frac{20}{9}$.

12. (4 points) Evaluate $\int \sin^4 x \cos^3 x \, dx$.

A. $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

B. $\frac{1}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

C. 0

D. $\frac{1}{7}\sin^7 x - \frac{1}{5}\sin^5 x + C$

E. $-\frac{1}{20}\cos^5 x \sin^4 x + C$

13. (4 points) Find the area under the curve $y = \frac{1}{x^2+4}$ between $x = -2$ and $x = 2$.

A. 2

B. $\frac{\pi}{4}$

C. 0

D. $\frac{\pi}{2}$

E. $\frac{\pi}{8}$

14. (4 points) Determine the values for A and B, such that $y = \dfrac{1}{x(10-x)} = \dfrac{A}{x} + \dfrac{B}{10-x}$.

A. $A = \frac{1}{2}, B = \frac{1}{2}$

B. $A = 10, B = \frac{1}{10}$

C. $A = 1, B = 1$

D. $A = \frac{1}{10}, B = \frac{1}{10}$

E. $A = \frac{1}{10}, B = 10$

15. (4 points) Which of the following integrals is NOT improper?

A. $\displaystyle\int_0^1 \frac{1}{x}\, dx$

B. $\displaystyle\int_0^1 \frac{1}{x^2-4}\, dx$

C. $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$

D. $\displaystyle\int_0^1 \frac{1}{x^2-1}\, dx$

E. $\displaystyle\int_0^1 \frac{1}{x-1}\, dx$

16. (4 points) Evaluate $\displaystyle\int_0^{\pi/2} x \cos x \, dx$.

A. $\frac{\pi}{2}$.

B. $\frac{\pi}{2} - 1$.

C. $\frac{\pi}{2} + 1$.

D. $-\frac{\pi}{2}$.

E. $-\frac{\pi}{2} + 1$.

More Challenging Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

17. (8 points) Show that $\displaystyle\int_m^{m^2} \frac{1}{x \ln x}\, dx = \ln 2$ for any $m > 1$.

Solution: Use $u-$substitution, with $u = \ln x$, $du = \dfrac{1}{x}dx$. Then

$$\begin{aligned} \int \frac{1}{x\ln x}\, dx &= \int \frac{1}{u}\, du \\ &= \ln|u| = \ln|\ln x| + C \end{aligned}$$

Therefore

$$\begin{aligned} \int_m^{m^2} \frac{1}{x\ln x}\, dx &= \ln|\ln x|\Big|_m^{m^2} \\ &= \ln|\ln m^2| - \ln|\ln m| \\ &= \ln\left|\frac{\ln m^2}{\ln m}\right| \\ &= \ln\left|\frac{2\ln m}{\ln m}\right| = \boxed{\ln 2} \end{aligned}$$

True/False questions. Circle the best answer. No work needed. No partial credit available.

18. (3 points) If $f(t) \leq g(t)$ and $\displaystyle\int_1^{\infty} g(t)\, dt$ converges, then $\displaystyle\int_1^{\infty} f(t)\, dt$ converges.

A. True

B. False

Solution: f may not be positive. Consider $g(t) = 1/x^2$ and $f(t) = -t$

19. (3 points) $\left(\dfrac{\log_2 x}{\log_5 x}\right)' = 0$.

A. True

B. False

Solution: Using the change of base formula we can see $\left(\dfrac{\log_2 x}{\log_5 x}\right)' = (\log_2(5))' = 0$

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	14	
3	14	
4	14	
5	14	
6	12	
7	12	
8	12	
9	14	
Total:	106	

No more than 100 points may be earned on the exam.

FORMULA SHEET

Integrals

- **Volume**: Suppose $A(x)$ is the cross-sectional area of the solid S perpendicular to the x-axis, then the volume of S is given by

$$V = \int_a^b A(x)\ dx$$

- **Work**: Suppose $f(x)$ is a force function. The work in moving an object from a to b is given by:

$$W = \int_a^b f(x)\ dx$$

- $\displaystyle\int \frac{1}{x}\ dx = \ln|x| + C$
- $\displaystyle\int \tan x\ dx = \ln|\sec x| + C$
- $\displaystyle\int \sec x\ dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int a^x\ dx = \frac{a^x}{\ln a} + C \qquad$ for $a \neq 1$
- **Integration by Parts**:

$$\int u\ dv = uv - \int v\ du$$

Derivatives

- $\dfrac{d}{dx}(\sinh x) = \cosh x \qquad \dfrac{d}{dx}(\cosh x) = \sinh x$
- Inverse Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

- If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Hyperbolic and Trig Identities

- Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh x}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh x}$$

$$\tanh(x) = \frac{\sinh x}{\cosh x} \qquad \coth(x) = \frac{\cosh x}{\sinh x}$$

- $\cosh^2 x - \sinh^2 x = 1$
- $\cos^2 x + \sin^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2 \sin x \cos x$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$