Basic Sets

Example 1. Let $S = \{1, \{2, 3\}, 4\}$. Indicate whether each statement is true or false.

(a) $|S| = 4$

(b) $\{1\} \in S$

(c) $\{2, 3\} \in S$

(d) $\{1, 4\} \subseteq S$

(e) $2 \in S$.

(f) $S = \{1, 4, \{2, 3\}\}$

(g) $\emptyset \subseteq S$

Example 2. Compute the cardinality of the set, E, where E is defined as

$$E = \{x \in \mathbb{R} \ : \ \sin(x) = 1/2 \text{ and } |x| \leq 5\}$$

Example 3. Suppose $A = \{0, 2, 4, 6, 8\}, B = \{1, 3, 5, 7\}$ and $C = \{2, 8, 4\}$. Find:
(a) $A \cup B$ (b) $A \setminus C$ (c) $B \setminus A$ (d) $B \cap C$ (e) $C \setminus B$

Example 4. Prove that $\{9^n : n \in \mathbb{Z}\} \subseteq \{3^n : n \in \mathbb{Z}\}$, but $\{9^n : n \in \mathbb{Z}\} \neq \{3^n : n \in \mathbb{Z}\}$.

Example 5. Prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$.

Functions

Example 6. For each of the following, determine the largest set $A \subseteq \mathbb{R}$, such that $f : A \to \mathbb{R}$ defines a function. Next, determine the range, $f(A) := \{y \in \mathbb{R} : f(x) = y, \text{ for some } x \in A\}$.

(a) $f(x) = 1 + x^2$,

(b) $f(x) = 1 - \frac{1}{x}$,

(c) $f(x) = \sqrt{3x - 1}$,

(d) $f(x) = x^3 - 8$,

(e) $f(x) = \frac{x}{x-3}$.

Injective, Surjective, Bijective Functions

Example 7. A function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f((m, n)) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.

Example 8. Define the operation

$$f(p) := \frac{d}{dx} p.$$

Does f define a function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f an injective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer. Is f a surjective function from $\mathbb{P}_4$ to $\mathbb{P}_4$? Justify your answer.

Example 9. Prove that the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

Example 10. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$ is injective.
Hint: A graph can help, but a graph is not a proof.

Example 11. Let $A = \mathbb{R} \setminus \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$.

(i) Prove that f is bijective.

(ii) Determine f^{-1}.

Example 12. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(x, y) = (2x - 3y, x + 1)$$

(a) Show that f is a bijection.

(b) Determine the inverse f^{-1} of f.

Example 13. Define the function, $f : \mathbb{P}_3 \to \mathbb{R}$ via the operation

$$f(p) := \int_0^1 p(x)dx.$$

Is f injective and or surjective from $\mathbb{P}_3$ to $\mathbb{R}$? Justify your answer.

Example 14. Consider the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse. Carefully justify that your answer does indeed yield the inverse function.

General Math Proofs

Example 15. Assume that you know that $x < y$. Carefully justify the statement that

$$x < \frac{x + y}{2} < y.$$

Example 16. Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Example 17. Show that there exists a positive even integer m such that for every positive integer n,

$$\left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{2}.$$

Example 18. Prove: For every real number $x \in [0, \pi/2]$, we have $\sin x + \cos x \geq 1$.

Example 19. Suppose $x, y \in \mathbb{R}^+$. Prove if $xy > 100$ then $x > 10$ or $y > 10$.

Logic

Example 20. For the sets $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P : A \subseteq B. \qquad Q : |A \setminus B| = 6.$$

Determine which of the following statements are true.
(a) $P \vee Q$ (b) $P \vee (\neg Q)$ (c) $P \wedge Q$ (d) $(\neg P) \wedge Q$ (e) $(\neg P) \vee (\neg Q)$.

Example 21. Let P : 15 is odd. and Q : 21 is prime. State each of the following in words, and determine whether they are true or false.
(a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\neg P) \vee Q$ (d) $P \wedge (\neg Q)$

Example 22. Rewrite the following using logical connectives and quantifiers

(a) If f is a polynomial and its degree is greater than 2, then f' is not constant.

(b) The number x is positive but the number y is not positive.

Example 23. Which of the following best identifies $f : \mathbb{R} \to \mathbb{R}$ as a constant function, where x and y are real numbers.

(a) $\exists x, \forall y, f(x) = y$.

(b) $\forall x, \exists y, f(x) = y$.

(c) $\exists y, \forall x, f(x) = y$.

(d) $\forall y, \exists x, f(x) = y$.

Example 24. Negate the following statements:

(a) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, x + y = 1$.

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = x$.

Example 25. In each of the following cases explain what is meant by the statement and decide whether it is true or false.

(a) $\lim_{x \to c} f(x) = L$ if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

(b) $\lim_{x \to c} f(x) = L$ if $\exists \delta > 0 \quad \forall \varepsilon > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

(c) $f : A \to B$ is surjective provided $\forall y \in B, \exists x \in A$ such that $f(x) = y$.

Induction

Example 26. If n is a non-negative integer, use mathematical induction to show that $5 \mid (n^5 - n)$.

Example 27. Prove by induction that $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$.

Example 28. Prove that if $n \in \mathbb{N}$, then $4^{2n} + 10n - 1$ is divisible by 25.

Even Odd Proofs

Example 29 (Prove using Contradiction)**.** Suppose $a \in \mathbb{Z}$. Prove that if a^2 is even, then a is even.

Example 30. Prove that If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Example 31. Let n be an integer. Show that n^2 is odd if and only if n is odd.

Example 32. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Example 33. Prove that there is no largest even integer.

Example 34. Prove the following claim

Claim: Suppose $a \in \mathbb{Z}$. If $a^2 - 2a + 7$ is even, then a is odd.

Real Analysis

Indexed Sets

Example 35. Let $B_1 = \{1, 2\}, B_2 = \{2, 3\}, \ldots, B_{10} = \{10, 11\}$; that is, $B_i = \{i, i+1\}$ for some $i = 1, 2, \ldots, 10$. Determine the following:

(i) $\bigcup_{i=1}^{5} B_i$

(ii) $\bigcup_{i=1}^{10} B_i$

(iii) $\bigcup_{i=3}^{7} B_i$

(iv) $\bigcup_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

(v) $\bigcap_{i=1}^{10} B_i$

(vi) $B_i \cap B_{i+1}$

(vii) $\bigcap_{i=j}^{j+1} B_i$, where $1 \leq j < 10$

(viii) $\bigcap_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

Example 36. Prove that $\bigcap_{x \in \mathbb{N}} [3 - (1/x)^2, 5 + (1/x)^2] = [3, 5]$.

Bounded and Unbounded Sets

Example 37. Discuss whether the following sets are bounded or not bounded.

(a) $A = \{-2, -1, 1/2\}$.

(b) $B = (-\infty, \sqrt{2})$.

(c) $C = \{1/2, 3/2, 5/2, 7/2, 9/2, \dots\} = \{\frac{2n-1}{2} | n \in \mathbb{N}\}$.

(d) $D = \left\{ \frac{(-1)^n}{n} \ : \ n \in \mathbb{N} \right\}$

(e) $E = \left\{ \frac{-1}{n} \ : \ n \in \mathbb{Q} \setminus \{0\} \right\}$

Sequences

Example 38. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{ \frac{3n+1}{7n-4} \right\}_{n \in \mathbb{N}}$

(b) $\left\{ \sin(\frac{n\pi}{4}) \right\}_{n \in \mathbb{N}}$

(c) $\{(1 + 1/n)^2\}_{n \in \mathbb{N}}$

(d) $\{(-1)^n n\}_{n \in \mathbb{N}}$

(e) $\left\{ \sqrt{n^2 + 1} - n \right\}_{n \in \mathbb{N}}$

Example 39. Using the definition of convergence, that is, an $\varepsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\lim_{n \to \infty} \frac{1}{n} = 0$.

(b) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$.

(c) $\lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2}$.

Example 40. For each of the following, determine whether or not they converge. If they converge, what is their limit? No proofs are necessary, but provide some algebraic justification.

(a) $\left\{ 3 + \frac{(-1)^n 2}{n} \right\}_{n \in \mathbb{N}}$

(b) $\left\{ \frac{n^2 - 2n + 1}{n-1} \right\}_{n \in \mathbb{N} \setminus \{1\}}$

(c) $\left\{ \frac{n}{n+1} \right\}_{n \in \mathbb{N}}$

Example 41. Using the definition of convergence, that is, an $\varepsilon - N$ argument, prove that the following sequences converge to the indicated number:

(a) $\lim_{n\to\infty}\left(3+\frac{2}{n^2}\right)=3.$

(b) $\lim_{n\to\infty}\frac{\sin(n)}{2n+1}=0.$

Example 42. Prove that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ does not converge.

Example 43. Use the formal definition of the limit of a sequence to prove that

$$\lim_{n\to\infty}\frac{2n-1}{3n+2}=\frac{2}{3}.$$

Open and Closed Sets

Example 44. Which of the following sets are open?

$$(i)(-3,3) \qquad (ii)(-4,5] \qquad (iii)(0,\infty) \qquad (iv)\{(x,y)\in\mathbb{R}^2 : y>0\}$$

$$(iv)\bigcup_{n=1}^{5}\left(-1+\frac{1}{n},1-\frac{1}{n}\right),\ \text{where } n\in\mathbb{N} \qquad (v)\{x\in\mathbb{R} : |x-1|<2\}.$$

Example 45. Let $A=[2,5]$. Discuss whether A^c an open set.

Example 46. Discuss which of the following sets are open.

(a) $(2,3)$.

(b) $(-4,8]$.

(c) $[1,3)$.

(d) $(-\infty,\infty)$.

(e) $[1,5]\cap[2,3]$.

Example 47. Which of the following sets are closed? Justify your answer.

$$(i)\ A=[2,5] \qquad (ii)\ B=(-1,0)\cup(0,1) \qquad (iii)\ C=\{x\in\mathbb{R} : |x-1|<2\}$$

$$(iv)\ D=\{-2,-1,0,1,2\} \qquad (v)\ \mathbb{Z}$$

Linear Algebra

Vector Spaces

Example 48. Show that the set $\mathbb{R}^2$ over $\mathbb{R}$ is not a vector space under the following definitions for vector addition and scalar multiplication:

$$x + y := (x_1 - y_1, x_2 - y_2)$$

and

$$\lambda x := (\lambda x_1, \lambda x_2),$$

where $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$.

Example 49. Under the usual matrix operations, is the set

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

a vector space over $\mathbb{R}$? Justify your answer.

Example 50. Define the set $V = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 \geq 2x_1 + 1\}$. Sketch a picture of the set V inside of the plane, $\mathbb{R}^2$. Is V a vector space over $\mathbb{R}$? Justify your answer!

Example 51. Define

$$V = \{p \in \mathbb{P}_2 \ : \ \forall x \in \mathbb{R}, \ p'(1) = 0\}.$$

Is V a vector space over $\mathbb{R}$?

Example 52. Find the additive inverse, in the vector space, of the following:

(a) In $\mathbb{P}_3$, of the element $-3 - 2x + x^2$.

(b) In the space of 2×2 matrices, of the element $\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$.

(c) In $\{ae^x + be^{-x} \,|\, a, b \in \mathbb{R}\}$, the space of functions of the real variable x, the element $3e^x - 2e^{-x}$.

You may assume that these vector spaces are defined over $\mathbb{R}$ and that in each case, natural definitions of addition and scalar multiplication hold.

Example 53. Show that the set of linear polynomials $\mathbb{P}_1 = \{a_0 + a_1 x \,|\, a_0, a_1 \in \mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations is a vector space over $\mathbb{R}$.

Linear Maps

Example 54. Define the function $A : \mathbb{P}^3 \to \mathbb{P}^5$ by

$$A(p)(x) = x^2 p(x) \qquad \text{for } x \in \mathbb{R}.$$

Is A a linear function? Justify your answer.

Example 55. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

Example 56. For each of the following L, answer "yes" or "no", and briefly justify your answer:

(a) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = \sin(x)$, a linear function?

(b) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = |x|^{1/2}$, a linear function?

(c) Is $L : \mathbb{R} \to \mathbb{R}$, with $L(x) = 51.5x$, a linear function?

(d) Is $L : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $L(p) = 3p$ a linear function? Find the images under L of $p, q \in \mathbb{P}_3$ defined by $p(x) = x^3 - 7$ and $q(x) = 2x^2 + 3x + 5$.

Example 57. Define the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the transformation

$$T\left((x_0, x_1)\right) = (x_0, 0),$$

where $(x_0, x_1) \in \mathbb{R}^2$. Is T a linear function? Justify your answer.

Abstract Algebra

Divisibility and Remainders

Example 58. Use the Euclidean Algorithm to find the greatest common divisor for each of the following pairs of integers:

(a) 51 and 288 $\quad$ (b) 357 and 629 $\quad$ (c) 180 and 252.

Example 59. Prove that the square of every odd integer is of the form $4k + 1$, where $k \in \mathbb{Z}$ (that is, for each odd integer $a \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $a^2 = 4k + 1$).

Example 60. Prove that if a divides b and c divides d, then ac divides bd.

Example 61. Answer true or false and give a complete justification. If p is prime, then $p^2 + 1$ is prime.

Example 62. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $c \mid b$ then $\gcd(a, c) = 1$. (Hint: Use proof by contradiction)

Equivalence Relations and Modular Arithmetic

Example 63. For $(a,b),(c,d) \in \mathbb{R}^2$ define $(a,b) \sim (c,d)$ to mean that $2a-b=2c-d$. Show that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

Example 64. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $4 \mid (x+3y)$. Prove $\sim$ is an equivalence relation. Describe its equivalence classes.

Example 65. Let $X = \mathbb{R}^2$, the xy-plane. Define $(x_1, y_1) \sim (x_2, y_2)$ to mean

$$x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Is $\sim$ an equivalence relation? Justify your answer. Give a geometric interpretation of the equivalence classes of $\sim$.

Example 66. Do the following calculations in $\mathbb{Z}_9$ (see page 238 of the text for a description of this notation), in each case expressing your answer as $[a]$ with $0 \leq a \leq 8$.

$$(a)\ [8]+[8] \qquad (b)\ [24]+[11] \qquad (c)\ [21]\cdot[15] \qquad (d)\ [8]\cdot[8].$$

Example 67. Let a and b be given integers. Prove $a \equiv b \mod 5$ if and only if $9a+b \equiv 0 \mod 5$.