August 2025 Algebra I Qualifying Exam

Problem 1. Let F be an algebraically closed field, V a nonzero finite-dimensional vector space over F, and $T : V \to V$ a linear transformation.

(a) Show that T is diagonalizable iff its minimal polynomial has no repeated roots.
Hint: Use the Structure Theorem for finitely generated $F[t]$-modules.

(b) Show that T is nilpotent iff its minimal polynomial is t^k, for some $k \in \mathbb{N}$.

Problem 2. Let $\mathrm{SL}_2(\mathbb{F}_p)$ be the group of 2×2 matrices with entries in the finite field $\mathbb{F}_p$ and determinant equal to 1.

(a) Find the size of $\mathrm{SL}_2(\mathbb{F}_p)$.

(b) Find a p-Sylow subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$.

(c) How many p-Sylow subgroups are there in $\mathrm{SL}_2(\mathbb{F}_p)$?

Problem 3. Let $\{1\} \to K \xrightarrow{i} G \xrightarrow{\pi} H \to \{1\}$ be a short exact sequence of finite groups. Show that if $P \subset G$ is a p-Sylow subgroup of G, then $\pi(P)$ is a p-Sylow subgroup of H.

Problem 4. Let R be a commutative ring, and let $f \in R$ be an element.

(a) Show that the localization R_f is isomorphic to $R[x]/(1 - xf)$ as a ring.

(b) Show that if $f = g + n$, where n is nilpotent, then R_f is isomorphic to R_g.

Problem 5. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module.

(a) State the definition of Noetherian R-module. Is $\mathbb{Q}$ Noetherian? Justify your answer.

(b) State the definition of a projective R-module. Is $\mathbb{Q}$ projective? Justify your answer.

Problem 6. Let R be a commutative ring, and let $P \subset R$ be a prime ideal.

(a) Show that if $I, J \subset R$ are ideals such that $I \cap J = P$, then $I = P$ or $J = P$.

(b) Show that R/P is an indecomposable R-module, meaning it is not isomorphic to a direct sum of (nonzero) R-modules.