ALGEBRA QUALIFYING EXAM I

(1) Let G be a group. In each of the following situations, either prove that G is finite, or give an example where G is not finite:
  (a) Every element of G has order 2.
  (b) G is generated by two elements $x, y \in G$, and both x and y have finite order.
  (c) G is generated by two elements x and y, and every element of G has order 2.

(2) Let p be a prime number. Let $G = \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ denote the group of invertible 2×2 matrices with coefficients in $\mathbb{Z}/p^2\mathbb{Z}$. Let $V = \mathbb{Z}/p^2\mathbb{Z}\times\mathbb{Z}/p^2\mathbb{Z}$, and let
$$\mathbb{P}(V) = \{W \subset V \mid V/W \cong \mathbb{Z}/p^2\mathbb{Z}\}$$
be the set of subgroups W of V such that V/W is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$.
  (a) Compute the order of G. (Hint: you may use the following fact without proof: the order of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is $(p^2-1)(p^2-p)$.)
  (b) Show that, for any $W \in \mathbb{P}(V)$, the group W is cyclic.
  (c) Show that G acts transitively on $\mathbb{P}(V)$ and compute the cardinality of $\mathbb{P}(V)$.
  (d) Let G act on $\mathbb{P}(V)\times\mathbb{P}(V)$ diagonally. Show that the map
$$\mathbb{P}(V)\times\mathbb{P}(V) \to \{1, p, p^2\}$$
given by $(W_1, W_2) \mapsto \#(W_1 \cap W_2)$ induces a bijection
$$G\backslash(\mathbb{P}(V)\times\mathbb{P}(V)) \longleftrightarrow \{1, p, p^2\}.$$

(3) Recall that a morphism $f : A \to B$ in a category is called an *epimorphism* if for any morphisms $g : B \to C$ and $h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.
  (a) Let R be a ring. Show that the epimorphisms in the category of R-modules are exactly the surjective R-module homomorphisms.
  (b) Show that the group homomorphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ given by $x \mapsto 2x$ is an epimorphism in the category of free abelian groups. (In the category of free abelian groups, objects are free abelian groups and morphisms are group homomorphisms.)
  (c) Show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category of rings.

(4) **This is a true/false question.** For each of the following statements, either prove the statement or give an example showing that the statement is false:
  (a) If R is a PID and $\mathfrak{p} \subset R$ is a prime ideal, then $R/\mathfrak{p}$ is a PID.
  (b) If R is a UFD and $\mathfrak{p} \subset R$ is a prime ideal, then $R/\mathfrak{p}$ is a UFD. (Hint: you may use the following fact without proof: if A is a UFD, then $A[x]$ is a UFD.)