ALGEBRA I QUALIFYING EXAM

AUGUST 2022

The exam will last 2 hours. You will be graded on 4 out of the 5 questions. If you submit solutions to all 5, you must state which question you choose for me to ignore on the first page of your solutions.

(1) Let p be a prime. Prove that there exists a non-abelian group of order p^3 with normal subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

(2) Find all isomorphism classes of abelian groups of order $10125 = 3^4 * 5^3$ with exactly 4 subgroups of order 15.

(3) Prove that there is no simple group of order 552.

(4) Recall that an element r of a ring R is nilpotent if $r^n = 0$ for some n.
   (a) Show that if R is commutative then the set of nilpotent elements form an ideal I called the nilradical of R.
   (b) Give a counterexample if R is not assumed to be commutative.
   (c) Give a commutative ring R for which the nilradical is not prime.

(5) (a) State the definition of a Noetherian module.
   (b) Let M be a module with submodule N. Show that M is Noetherian if and only if N and M/N are both Noetherian.
   (c) Let $\mathcal{F}$ be the ring of real valued functions on the interval $[0, 1]$. Which of the following $\mathcal{F}$-modules are Noetherian? Justify your answers:
      (i) $\mathcal{M}_1 := \{f : [0, 1] \to \mathbb{R} \mid f(a) = 0 \text{ for } a \notin \{0, 1/2, 1\}\}$
      (ii) $\mathcal{M}_2 := \{f : [0, 1] \to \mathbb{R} \mid f(a) = 0 \text{ for all but finitely many } a \in [0, 1]\}$
      (iii) $\mathcal{F}$ as a module over itself.