ALGEBRA I QUALIFYING EXAM AUGUST 2024

(1) Let H be a proper subgroup of a finite group G.

  (a) Prove that for any $g \in G$, the subgroup gHg^{-1} of G is isomorphic to H.

  (b) Prove that as g ranges over the elements of G, the number of distinct sets gHg^{-1} is at most the index of H in G.

  (c) Conclude that $G \neq \bigcup_{g \in G} gHg^{-1}$.

(2) Prove that there is no simple group of order 280.

(3) Let R be a commutative ring and P, Q prime ideals in R. Prove that $\mathrm{Hom}_R(R/P, R/Q)$ is nonzero if and only if $P \subseteq Q$.

(4) (a) Let S be a commutative ring with identity and τ a unital ring automorphism of S. Show that the image of a principal ideal of S under τ is principal and that the image of a prime ideal is prime.

  (b) Consider the principal ideal $I = (y^2 - x^3 + x)$ in $\mathbb{Q}[x, y]$ and let $R = \mathbb{Q}[x, y]/I$.

    (i) Show that the automorphism of $\mathbb{Q}[x, y]$ given by $x \mapsto x$ and $y \mapsto -y$ descends to an automorphism σ of R.

    (ii) Show that the ideal $P = (x, y)R$ in R is a prime ideal.

    (iii) Show that P^2 is a principal ideal.

(5) Let R be a commutative ring with identity.

  (a) Prove that a module M over R is projective if and only if M is a direct summand of a free module.

  (b) Prove that a projective module over an integral domain is torsion free.

  (c) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely-generated R-modules. Either prove or give a counter-example to the following.

    (i) If M_1 and M_2 are projective, then so is M_3.

    (ii) If M_2 and M_3 are projective, then so is M_1.

(6) Let $\mathbb{F}_2$ be the field with 2 elements and consider the ring $R = \mathbb{F}_2[x]$. List, up to isomorphism, all R-modules with 8 elements.