ALGEBRA I QUALIFYING EXAM JANUARY 2024

(1) Let G be a non-trivial group such that G contains a proper subgroup H which contains every proper subgroup of G. Show that G is cyclic of order p^i for some prime p.

(2) Let G be a free abelian group of rank r. Show that G has only finitely many subgroups of a given finite index n.

(3) (a) Suppose that R is a commutative ring with identity. Suppose that $a \in R$ is not nilpotent (meaning there is no integer n such that $a^n = 0$). Prove that there is a prime ideal in R that does not contain a.
Hint: Consider the set of ideals in R that do not contain any power of a

(b) Use the above to show that the set of all nilpotent elements of R is the intersection of all prime ideals of R.

(4) Let I be an ideal of a commutative ring R and $a \in R$. Consider the ideals $I + Ra$ and $(I : a) = \{x \in R \mid ax \in I\}$.

(a) Show that $(I + Ra)/Ra \cong I/a(I : a)$.

(b) Use the above to show that if $I + Ra$ and $(I : a)$ are both finitely-generated, then I is finitely-generated.
Hint: Five Lemma

(5) Let R be a PID and K its field of fractions.

(a) Show that any subring L of $M_n(K)$ which is finitely-generated as a (unitary) R-module, must be free as an R-module.

(b) A subset S of K has a **common denominator** in R when there is a nonzero $r \in R$ such that $rS \subset R$.

Let H be a subgroup of $GL_n(K)$ whose matrix entries have a common denominator in R.

(i) Consider the subset of K^n given by $M = \sum_{h \in H} h(R^n) \subset K^n$, which consists of finite sums of vectors belonging to some $h(R^n) = \{h(r_1, \ldots, r_n) \mid r_1, \ldots, r_n \in R\}$. Prove that $M \cong R^n$ as an R-module.

(ii) Use Part (i) to show that H is conjugate to a subgroup of $GL_n(K)$ with matrix entries in R.
Hint: If $e_1, \ldots, e_n$ is the standard basis of R^n, then any matrix A in $GL_n(K)$ has i-th column given by $A(e_i) \subset K^n$.

(6) For commutative rings A, B with identity, a ring homomorphism $A \to B$, and non-zero A-modules M, N consider the canonical map

$$B \otimes_A \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$$

given by

$$b \otimes f \mapsto (x \otimes m \mapsto bx \otimes f(m)).$$

(a) Find examples of A, B, $A \to B$, M, and N such that the above map is the zero map.

(b) Prove that if M is a finitely-generated projective A-module, then the above map is an isomorphism.