January 2025 Algebra Qualifying Exam

Problem 1. Let G be a group of size $385 = 5 \times 7 \times 11$.

(a) Show that G contains normal subgroups of sizes 7 and 11, and the subgroup of size 7 is contained in the center $Z(G)$.

(b) Show that there are only two possibilities for the group G up to isomorphism.

Problem 2. Let R be a $\mathbb{Q}$-algebra that is finite-dimensional as a $\mathbb{Q}$-vector space.

(a) Show that if $rs = 1$, then $sr = 1$ in R.

(b) Show that if for any nonzero $x, y \in R$, we have $xy \neq 0$, then R is a division ring. (That is, every nonzero element has a multiplicative inverse.)

Problem 3. Let R be a commutative ring, and let M be an R-module. Show that $M \simeq (0)$ if and only if $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ in R. Recall that the notation $M_\mathfrak{m}$ means the localization of M with respect to the multiplicative set $R \setminus \mathfrak{m}$.

Problem 4. Let G be a simple group, and let $H \subset G$ be a non-trivial proper subgroup.

(a) Show that the subgroup $\tilde{H}$ generated by all the conjugate subgroups gHg^{-1}, for $g \in G$, is equal to G itself.

(b) Show that the union of all the conjugate subgroups gHg^{-1}, for $g \in G$, is not a subgroup, assuming that G is finite.

(c) Find a counterexample to part (b) assuming that G is infinite. Hint: Matrix groups. No justification is required for the counterexample.

Problem 5. Let R be a commutative ring, and let $x \in R$ be an idempotent element ($x^2 = x$) with $x \neq 0, 1$. Show that $R \simeq R_1 \times R_2$ for some nonzero rings R_1 and R_2.

Problem 6. Let R be a commutative ring, and let M be an R-module. For a given submodule $N \subset M$, let

$$\mathrm{Ann}(N) = \{r \in R : r \cdot n = 0 \text{ for all } n \in N\}.$$

(a) Show that $\mathrm{Ann}(N) \subset R$ is an ideal.

(b) Let V be a finite-dimensional F-vector space, and let $T : V \to V$ be a F-linear map. Viewing V as an $F[t]$-module in the usual way, show that for any polynomial $q(t) \in F[t]$ dividing the minimal polynomial of T, there exists an $F[t]$-submodule $W \subset V$ such that $\mathrm{Ann}(W) = (q(t))$.