Algebra I Qualifying Exam
January 2026

Problem	**Score**
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

Name: ______________________________

Exam instructions:

- All work must be shown, and all answers justified appropriately. Cross out anything you do not want graded.
- Feel free to use the backs of pages if you need more space. Clearly label and include any scratch paper you want graded as part of an answer.
- You may ask proctors questions to clarify problems on the exam.
- Notes, AI's, books, phones, etc are not allowed.

This is a timed exam, so prioritize getting your ideas clearly laid out on the page first. You can label statements you need but are not sure about as claims, use them, then come back and prove them later once you have tried every problem.

Problem 1. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{R},\ a > 0 \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| \, c \in \mathbb{R} \right\}.$$

You may assume without proof that these are both groups under matrix multiplication.

(a) Show that N is normal in G.

(b) Prove that G/N is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}_{>0}$.

(c) Does there exist a group H such that $N < H < G$ (all distinct)? Prove or disprove.

Problem 2. Let N be the subgroup of $\mathbb{Z}^3$ generated by $(4, 6, 4)$, $(12, 12, 8)$, and $(10, 6, 8)$.

(a) Compute the Smith Normal Form of the associated matrix.

(b) Determine both the invariant factor decomposition and the elementary divisor decomposition of N.

(c) Determine both the invariant factor decomposition and the elementary divisor decomposition of the quotient group $\mathbb{Z}^3/N$.

Problem 3. Suppose that R is an integral domain and

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x] \quad \text{with } n \geq 1.$$

Eisenstein's Criterion states: If $p \in R$ is a prime element with $p \mid a_i$ for $0 \leq i \leq n-1$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible in $R[x]$.

Prove Eisenstein's Criterion.

Problem 4. Let F be a field, and let $R = F[x, y]$ be the ring of polynomials in two variables with coefficients in F.

(a) Show that $M = (x - 3,\, y + 1)$ is a maximal ideal of R.

(b) Prove that $P = (x^2 - y)$ is a prime ideal of R.

(c) Is P a maximal ideal of R? Justify your answer.

Problem 5. Suppose that R is a commutative (unital) ring. Let M be an R-module, and let $N \subseteq M$ be an R-submodule. Show that if both N and M/N are finitely generated, then M is finitely generated.

Problem 6. Let G be a group with 21 elements.

(a) Show that G has a normal subgroup of size 7.

(b) Show that if G is abelian, then G is cyclic.

(c) Exhibit a nonabelian group of order 21.