August 2025 Algebra II Qualifying Exam

Problem 1. Let p be a prime, and let $\zeta_n = e^{2\pi i/n}$ for $n \in \mathbb{N}$.

(a) Describe the Galois group of the extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_p)$. For which values of p does this Galois group have a quotient isomorphic to $\mathbb{Z}/5\mathbb{Z}$?

(b) Show that there exists an extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 2. Let $F \subset K$ be a Galois extension with Galois group G. Recall that the trace map $T : K \to K$ can be defined as

$$T(\alpha) := \sum_{g \in G} g(\alpha).$$

(a) Show that T is an F-linear map, and it takes values in F, so we can write $T : K \to F$.

(b) Using independence of characters, show that there exists $c \in K$ such that $T(c) \neq 0$.

(c) Assume that $G \simeq C_n$, generated by an element $g \in G$. Show that if $T(\alpha) = 0$, then

$$\beta := \frac{1}{T(c)}(\alpha g(c) + (\alpha + g(\alpha))g^2(c) + \cdots + (\alpha + g(\alpha) + \cdots + g^{n-2}(\alpha))g^{n-1}(c))$$

satisfies the property that $\alpha = \beta - g(\beta)$.

Problem 3. Let F be a field of characteristic $p > 0$, $c \in F$, and consider the polynomial

$$f(x) = x^p - x + c \in F[x].$$

(a) Show that if $f(x)$ is irreducible, then $K = F[x]/(x^p - x + c)$ is a Galois extension of F with Galois group isomorphic to C_p.

(b) Show conversely that if $F \subset K$ is a Galois extension with Galois group isomorphic to C_p, then it must be isomorphic to the example in part (a).

Hint: Observe that $-1 \in K$ has trace 0, and then use the previous problem.

Problem 4. Describe all abelian groups A which can occur in a short exact sequence

$$0 \to \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to A \to \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \to 0.$$

Problem 5. Describe all finite-dimensional complex irreducible representations of D_4, the dihedral group of order 8.

Problem 6. Let $R = F[x, y]$, and let $I \subset R$ be the ideal generated by x and y.

(a) Compute $\mathrm{Ext}^i_R(I, R/I)$ for all $i \geq 0$, using a projective resolution.

(b) Show that $\mathrm{Ext}^1_R(I, R/I) \simeq \mathrm{Ext}^2_R(R/I, R/I)$, using a long exact sequence.