ALGEBRA II QUALIFYING EXAM

AUGUST 2022

The exam will last 2 hours. You will be graded on 4 out of the 5 questions. If you submit solutions to more than 4, you must state which question you choose for me to ignore on the first page of your solutions.

(1) Let $F = \mathbb{Q}(\sqrt[4]{3}, i)$.
  (a) Identify isomorphism class of the Galois group G of $F/\mathbb{Q}$ in terms of the classification of small groups.
  (b) Find all intermediate fields $F/K/\mathbb{Q}$ with $[K : \mathbb{Q}] = 2$.

(2) Let F be a field of characteristic $p > 0$, such that $f(X) = X^p - X + a \in F[X]$ does not have a root in F. Show that $f(X)$ is irreducible, and that the Galois group of f is cyclic of order p.
Hint: if α is a root of f in an extension of F, show that $\alpha + 1$ is also a root of f.

(3) (a) Show that if M and N are projective R-modules, then $M \otimes_R N$ is projective.
  (b) Show that if M and N are injective $\mathbb{Z}$-modules then $M \otimes_{\mathbb{Z}} N$ is injective.
  Hint: This does not hold if $\mathbb{Z}$ were replaced by a general ring R.

(4) Show that $\mathrm{Tor}_i^{\mathbb{Z}/p^3\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for all $i \geq 0$.

(5) There is a group of order 24 called $\mathrm{SL}_2(\mathbb{F}_3)$. The table below contains the sizes of the conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_3)$, and two of its irreducible complex characters. Use these characters to compute the entire complex character table of $\mathrm{SL}_2(\mathbb{F}_3)$.

|ccl(g)|	1	1	6	4	4	4	4
$\chi_2(g)$	1	1	1	ξ	ξ^2	ξ^2	ξ
$\chi_4(g)$	2	-2	0	-1	-1	1	1

Here ξ is a primitive cube root of unity.