ALGEBRA II QUALIFYING EXAM AUGUST 2024

(1) (a) Show that the quotient group $\mathbb{Q}/\mathbb{Z}$ is an injective object in the category of abelian groups.

(b) Show that there are no nonzero projective objects in the category of torsion abelian groups.
Hint: Use the previous part of the problem

(2) Let G be an infinite cyclic group.

(a) Show that $\mathbb{Z}[G]$ is isomorphic to the ring $\mathbb{Z}[x, x^{-1}]$.

(b) Show that the sequence

$$0 \to \mathbb{Z}[x, x^{-1}] \xrightarrow{\delta} \mathbb{Z}[x, x^{-1}] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\delta(f(x)) = (x-1)f(x)$ and $\varepsilon(f(x)) = f(1)$, is a free resolution of $\mathbb{Z}$.

(c) Let M be a G-module. Show that

(i) $H^0(G, M) = M^G$, where M^G denotes the G-invariant submodule of M,

(ii) $H^1(G, M) \cong M/(g-1)M$, where g is a generator of G,

(iii) $H^i(G, M) = 0$ for $i \geq 2$.

(3) Let p be a prime and ζ_p a primitive p-th root of unity. For a positive integer d, show that the field $\mathbb{Q}(\zeta_p)$ has a subfield F such that $[F : \mathbb{Q}] = d$ if and only if $p \equiv 1 \mod d$.

(4) Let p be a prime number and $\mathbb{F}_p$ the finite field with p elements. Let $K = \mathbb{F}_p(t)$ be the field of rational functions in the variable t over $\mathbb{F}_p$.

(a) Show that the splitting field of the polynomial $x^p - t \in K[x]$ over K is inseparable.

(b) Show that the splitting field of $x^p - t - 1 \in K[x]$ over K is the same as the splitting field of $x^p - t$.

(c) Show that K has a single degree p inseparable extension.
Hint: first show that an inseparable degree p extension of K must be of the form $K(\alpha)$ for some inseparable element α and use what you know about the form that inseparable polynomials must have.

(5) Let p be a prime and G the multiplicative group of of 3×3 matrices over $\mathbb{F}_p$ which are upper triangular and have 1s on the diagonal. Let V be the space of complex functions on $\mathbb{F}_p$.

For any $z \in \mathbb{C}$ such that $z^p = 1$, define a representation R_z of the group G on V by

$$R_z \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f(x) = f(x-1), \quad R_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} f(x) = z^x f(x).$$

(a) Show that R_z is a representation and compute $R_z(g)$ for all $g \in G$.

(b) Show that R_z is irreducible if and only if $z \neq 1$.

(c) Classify all 1-dimensional representations of G and show that R_1 decomposes into a direct sum of 1-dimensional representations, where each occurs exactly once.

Hint: For the first part, consider the action of $R_z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *on eigenvectors*

(d) Use the previous parts to classify all irreducible representations of G.