ALGEBRA II QUALIFYING EXAM

You will be graded on 4 out of 5 questions. If you submit solutions to all 5 please specify on the first page which question you choose to be ignored.

(1) Let $\alpha = \sqrt{2} + \sqrt{5}$.
  (a) Find the minimal polynomial f of α over $\mathbb{Q}$.
  (b) Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ and hence that $\mathbb{Q}(\alpha)$ is the splitting field of f over $\mathbb{Q}$.
  (c) Find the other roots of f, and write them in terms of α.

(2) Explicitly find all the intermediate fields of the extension $\mathbb{Q}(\xi_8)/\mathbb{Q}$ where ξ_8 is a primitive 8th root of unity.

(3) Show directly (i.e. without appealing to the fact that A is injective) that if A is a divisible $\mathbb{Z}$-module, and M is any $\mathbb{Z}$-module, then a short exact sequence

$$0 \to A \to M \to \mathbb{Z}/n\mathbb{Z} \to 0$$

splits.

(4) Let R be the ring $\mathbb{C}[x]/(x^n)$ for some $n \geq 2$, and view $\mathbb{C}$ as an R-module via the quotient $\mathbb{C} \cong R/(x)$.
  (a) Show that

$$\cdots \xrightarrow{f \mapsto x^{n-1} f} \mathbb{C}[x]/(x^n) \xrightarrow{f \mapsto xf} \mathbb{C}[x]/(x^n) \xrightarrow{f \mapsto x^{n-1} f} \mathbb{C}[x]/(x^n) \xrightarrow{f \mapsto xf} \mathbb{C}[x]/(x^n) \xrightarrow{/(x)} \mathbb{C}$$

  is a projective resolution of $\mathbb{C}$ as an R-module, where $\ldots$ indicates the the sequence continues infinitely in the same manner.
  (b) For all $i \geq 0$, compute the dimension of $\mathrm{Tor}_i^R(\mathbb{C}, \mathbb{C}[x]/(x^2))$ as a complex vector space.

(5) Show that if R is a semi-simple ring, and M is a simple R-module, then R has a submodule isomorphic to M.