Note: All statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it is a well-known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

1. Compute the following integral
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\, dx.$$

2. Consider the polynomial
$$p(z) := z^6 + az^2 + 1$$
Find an $a > 0$ so that $p(z)$ has exactly 4 zeros in the annulus $\{z \in \mathbb{C}\colon \frac{1}{2} < |z| < 2\}$.

3. Let f be an entire function with finitely many zeros, and suppose there exists constants $0 < \rho < 1$ and $A, B > 0$ so that
$$|f(z)| \leq Ae^{B|z|^\rho}$$
for all $z \in \mathbb{C}$. Show that f is a polynomial.

4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of entire functions which is uniformly bounded on compact subsets. Suppose that there exists a polynomial $p(z)$ of degree d such that for each $z \in \mathbb{C}$, $(f_n(z))_{n\in\mathbb{N}}$ converges to to $p(z)$. Show that there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $f_n(z)$ has at least d zeros (counting multiplicities).

5. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Suppose that $f\colon \Omega \to \Omega$ is analytic and that for some $z_0 \in \Omega$ we have $f(z_0) = z_0$ and $f'(z_0) = 1$. Show that $f(z) = z$.