Qualifying Exam   Complex Analysis   August, 2024

Instructions: You may freely use any results covered in the lectures or homework, but must clearly specify what results you apply. You must show all necessary work, justify your claims, and ensure your answer is in the simplest form. Recall that $\mathbb{D}$ represents the unit disk $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$, and $\deg(P)$ is the degree of (a polynomial) P.

1. Let $R > r > 0$. Compute
$$\int_0^{2\pi} \frac{e^{in\theta}}{R^2 + r^2 + 2Rr\cos\theta}\, d\theta.$$
for all $n \in \mathbb{Z}$.

2. Let P and Q be two complex polynomials, which are not constant zero. Suppose P and Q have no common zeros, and $\deg(Q) < \deg(P) = m \in \mathbb{N}$. Let $R = \frac{P}{Q}$ be a rational function. Prove that there is a finite set $S \subset \mathbb{C}$ such that for any $w \in \mathbb{C} \setminus S$, the set $R^{-1}(\{w\})$ contains exactly m elements (without counting multiplicities).

   Hint: Recall the Fundamental Theorem of Algebra and the fact that a zero z_0 of a holomorphic function f is not simple if and only if $f'(z_0) = 0$.

3. Let h be harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose $h(e^{i\theta}) = e^{-\sin\theta}\cos(\cos\theta)$ for $\theta \in \mathbb{R}$. Find the value of $h(1/2)$. Justify your answer.

   Hint: Find an analytic function whose real part agrees with h on $\partial\mathbb{D}$.

4. Find the radius of convergence of the Taylor series of
$$f(z) = \frac{1}{1 + z + z^2 + z^3 + z^4 + z^5}$$
centered at 1.

5. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $f(0) = f'(0) = 0$. Prove that $|f(z)| \le |z|^2$ for all $z \in \mathbb{D}$.

   Hint: Consider $h(z) = f(z)/z^2$ for $z \in \mathbb{D} \setminus \{0\}$.