Complex Analysis qualification exam: August 2023

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Suppose that $f(1/n) = e^{-n}$ for all $n \in \mathbb{N}$. Show that f cannot be holomorphic in a neighborhood of 0.

(2) For $a > 0$, evaluate

$$\int_0^{+\infty} \frac{\cos(ax)\,dx}{(1+x^2)^2}.$$

(3) Let f be an entire function satisfying $|f(z)| \geqslant 1$ for all $z \in \mathbb{C}$ with $|z| \geqslant 10$. Show that f is a polynomial.

(4) Let

$$U = \{z \in \mathbb{C} \colon |z| < 1\} \setminus [0,1].$$

In other words, U is the open unit disk with the line segment between 0 and 1 removed. Find a conformal mapping between U and the upper half plane.

(5) Show that there does not exist a function f holomorphic in $\{z \in \mathbb{C} \colon |z| > 3\}$ with

$$f'(z) = \frac{z^2+1}{z(z-1)(z-2)}$$

for all z from the above set.