Qualifying Exam   Complex Analysis   May, 2024

Instructions: You may freely use any results covered in the lectures or homework, but must clearly specify which results you apply. You must show all necessary work, justify your claims, and ensure your answer is in the simplest form. Recall that we use the following symbols: $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $D(w,r) = \{z \in \mathbb{C} : |z-w| < r\}$, $\overline{D}(w,r) = \{z \in \mathbb{C} : |z-w| \leq r\}$, and $\mathbb{D} = D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

1. Let f be a holomorphic function defined on a convex domain D. Suppose $\mathrm{Re}(f'(z)) > 0$ for all $z \in D$. Prove that f is injective.

2. Let $R > r > 0$. Compute

$$\int_0^{2\pi} \log(R^2 + r^2 + 2Rr\cos\theta)d\theta.$$

   Here the log is the real logarithm function.

3. Let f be an analytic function from $\mathbb{D}$ to $\mathbb{H}$. Prove that for any $w \in \mathbb{D}$,

$$|f'(w)| \leq \frac{2\,\mathrm{Im}(f(w))}{1-|w|^2}.$$

4. Let f be a meromorphic function on $\mathbb{C}$, which has finitely many poles and satisfies that $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a rational function, i.e., the quotient of two polynomials.

5. Find an injective analytic function from $U := \mathbb{D} \setminus (\overline{D}(1+i,1) \cup \overline{D}(1-i,1))$ (illustrated below) onto $\mathbb{H}$. You may express your function as a composition of several functions, each of which should have an explicit formula.