Complex Analysis qualification exam: May 2023

Note: all statements require proofs. You can make references to standard theorems from the course; however, you need state the relevant part of the theorem in your own words, unless it's a well known named theorem. For example, "we had a theorem in the class that said that any continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous" is a good reference, and "by the uniqueness theorem from the class, f is unique" is not a good reference.

(1) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is totally differentiable at z_0 and the limit
$$\lim_{z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$$
exists. Show that either f or $\bar{f}$ is complex differentiable at z_0.

(2) Let $\gamma\colon [0,1] \to \mathbb{C} \setminus \{i, -i\}$ be a continuous path with $\gamma(0) = 0$, $\gamma(1) = 1$. Show that
$$\int_\gamma \frac{dz}{1+z^2} \in \frac{\pi}{4} + 2\pi i \mathbb{Z}.$$

(3) For $R > 0$, find a conformal isomorphism between
$$\mathbb{C} \setminus ((-\infty, -R) \cup (R, +\infty))$$
and the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$.

(4) Calculate the following integral for all values of $N \in \mathbb{N}$:
$$\oint_{|z|=\pi N+1/2} \frac{dz}{\sin z},$$
where the circle $|z| = \pi N + 1/2$ is parametrized counter-clockwise.

(5) Show that for every $\rho \in (0,1)$ there exists $N = N(\rho)$ such that the function
$$f_n(z) = 1 + z + 2z + 3z^2 + 4z^3 + \dots$$
does not have any zeros in $\{z \in \mathbb{C}\colon |z| < \rho\}$ for $n > N(\rho)$.