# Geometry Qualifying Exam

Michigan State University – August 2019

**Instructions.** Solve any **5 of the following 6 problems. Show clearly which 5 problems you want graded.** You must justify your claims either by direct arguments or by referring to specific theorems.

*All manifolds, functions, vector fields, etc. are assumed to be smooth.*

**Problem 1.** Let $S$ denote the unit sphere in $R^n$.

(a) Describe the tangent bundle $TS$ as a subset of $\mathbb{R}^{2n}$.

(b) Using (a), prove that $TS$ is a submanifold of $\mathbb{R}^{2n}$.

(c) Using (a), show that $\pi : TS \to S$ is a vector bundle.

**Problem 2.** Let $\omega$ be a 1-form on a manifold $M$, and let $\text{Vect}(M)$ denote the space of all smooth vector fields on $M$. Define

$$\eta : \text{Vect}(M) \times \text{Vect}(M) \to C^\infty(M)$$

by

$$\eta(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \tag{1}$$

for all vector fields $X, Y$ on $M$. Prove that $\eta = d\omega$ as follows:

(a) Show that $\eta$ is bilinear and skew-symmetric in $X$ and $Y$.

(b) Show that $\eta(fX, Y) = f\eta(X, Y)$ for all $f \in C^\infty(M)$.

Note: Together, (a) and (b) imply that $\eta$ is a 2-form.

(c) Prove that $d\omega(X, Y) = \eta(X, Y)$ in the case where $X$ and $Y$ are basis vector fields $\frac{\partial}{\partial x^i}$ in a local coordinates $\{x^i\}$.

(c) Explain why (c) implies that $\eta = d\omega$.

**Problem 3.** Let $G$ be a Lie group. For each $g \in G$, let $L_g : G \to G$ be left multiplication by $g$ (defined by $L_g(h) = gh$ for all $h \in G$) and set

$$\mathfrak{g} = \{ \text{ all left-invariant vector fields on } G \}.$$

(a) Show that $\mathfrak{g}$ is a vector space.

(b) Prove that there is a isomorphism $\mathfrak{g} \cong T_eG$, where $T_eG$ is the tangent space at the identity element $e \in G$.

**Problem 4.** Let $\omega$ be a smooth 1-form on a manifold $M$ such that $\int_C \omega = 0$ for every sufficiently small closed curve $C$ (i.e. every closed curve of length less that some fixed $\varepsilon$). Prove that $\omega$ is closed.

*Hint:* Suppose that $(d\omega)(p) \neq 0$ at some $p \in M$.

**Problem 5.** Consider the map $f : S^2 \to \mathbb{R}^6$ defined on the unit sphere $S^2 \subset \mathbb{R}^3$ by

$$f(x, y, z) = \left(x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}xz, \sqrt{2}xy\right).$$

(a) Write down the differential $Df$ as a matrix.

(b) Prove that $f$ is an immersion.

(c) Prove that $f$ is *locally* an embedding.

(d) Show that $f$ is not injective. Relate this to $\mathbb{RP}^2$.

**Problem 6.** Compute $\int_M \alpha$ where $\alpha$ is the 2-form

$$\alpha = 3x^2\, dx \wedge dy + x^2(3y^2 + 1)\, dy \wedge dz + 2xy\, dx \wedge dz$$

and $M$ is the ellipsoidal surface

$$M = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + 4z^2 = 1\right\}$$

with the "outward normal' orientation.

*Hints:* (i) Simplify the integral by noting that $\alpha \doteq \beta + d\gamma$ where $\gamma = x^3\, dy + x^2 y\, dz$.

(ii) Then parameterize the upper half $M^+$ of $M$ as a graph over the unit disk in the $(x, y)$-plane, and use polar coordinates.