Geometry Qualifying Examination

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Instructions: Solve 4 out of the 5 problems. You must justify all your claims either by direct arguments or by referring to well known and basic theorems. Indicate clearly which four problems you would like us to grade.

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

(1) Show that the set $\text{graph} f = \{(x, f(x))\} : x \in \mathbb{R}^n\}$ is a smooth submanifold of $\mathbb{R}^{n+1}$.

(2) State the regular value theorem.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x, y) = x^3 + xy + y^3.$$

What are the regular values of f?

Problem 2.

(1) State the Stokes theorem. You must explain the meaning of the symbols involved in the formula.

(2) Let $f_0 : M^m \to X^n$ and $f_1 : M^m \to X^n$ be smooth maps between smooth manifolds. Assume that M^m is closed and oriented and the two maps are smoothly homotopic, i.e. there exist a smooth map $F : M^m \times [0, 1] \to X^n$ s.t. $f_0 = F(\cdot, 0), f_1 = F(\cdot, 1)$. Prove that if $\omega \in \Omega^m(X)$ is a closed m-form on X, then

$$\int_{M^m} f_0^* \omega = \int_{M^m} f_1^* \omega.$$

Problem 3. On $\mathbb{R}^3 - \{0\}$ consider the differential form

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

(1) Show that ω is closed.

(2) Let $\Sigma \subset \mathbb{R}^3 - \{0\}$ be a connected and closed (compact and without boundary) surface. Σ bounds in $\mathbb{R}^3$ a bounded domain U and is given the induced boundary orientation. Prove that

$$\frac{1}{4\pi} \int_{\Sigma} \omega = \begin{cases} 1, & \text{if } 0 \in U; \\ 0, & \text{if } 0 \notin U. \end{cases}$$

Problem 4. Let M^3 be a 3-manifold and let θ be a smooth 1-form on M^3 such that $\theta \wedge d\theta$ is a volume form. Prove that there does not exist an embedding $f : \Sigma^2 \to M^3$ with Σ^2 a 2-manifold such that $f^*\theta = 0$.

Problem 5. Let M be a smooth n-dimensional manifold. Let $X_1, \ldots, X_n$ be smooth, linearly independent vector fields on an open set $U \subset M$. Then there are smooth functions c_{ij}^k such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

(1) Prove the following identities

$$c_{ij}^k = -c_{ji}^k,$$

$$\sum_{l=1}^{n} c_{ij}^l c_{kl}^p + c_{jk}^l c_{il}^p + c_{ki}^l c_{jl}^p = 0.$$

(2) Let $\omega^1, \ldots, \omega^n$ be the dual basis of 1-forms, i.e. $\omega^i(X_j) = \delta_j^i$. Prove

$$d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{n} c_{ij}^k \omega^i \wedge \omega^j.$$