Geometry Qualifying Examination

Aug. 22, 2024

Instructions: Solve 5 out of the 6 problems. You must justify all your claims either by direct arguments or by referring to well known and basic theorems. Indicate clearly which 5 problems you would like us to grade.

Problem 1. Let $GL(n,\mathbb{R}) = \{A : A \text{ is a real invertible } n \times n \text{ matrix}\}$ and let $O(n) = \{A : A \text{ is a real } n \times n \text{ matrix satisfying } AA^t = I\}$

(a) Show that $GL(n,\mathbb{R})$ is a smooth manifold of dimension n^2.

(b) Show that $O(n)$ is a smooth submanifold of $GL(n,\mathbb{R})$. What is its dimension? Prove your claim.

(d) Find the tangent space at the identity of $GL(n,\mathbb{R})$ and $O(n)$.

Problem 2. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x, y, z) = x^2 + 2xy + 4y^2 + z^4$.

(1) Prove that the level set $\Sigma := F^{-1}\{1\}$ is a submanifold of $\mathbb{R}^3$.

(2) Is Σ compact? Prove your claim.

(3) Find the critical points of the function $f(x, y, z) = y$ on Σ.

Problem 3. Consider $\mathbb{R}^4$ with coordinates (x, y, z, w). Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $f(u, v) = (v, u, v^3, u^2 v)$. Define a 2-form on $\mathbb{R}^4$ by

$$\alpha = w^2 dy \wedge dz + y^3 dy \wedge dw - 2yw dz \wedge dw.$$

(1) compute $f^*\alpha$.

(2) show $d\alpha = 0$.

(3) Find a 1-form β on $\mathbb{R}^4$ such that $\alpha = d\beta$.

Problem 4.

(1) What is an orientation on a finite dimensional real vector space V? If $\dim V = 3$ and $\{e_1, e_2, e_3\}$ is a basis, determine if $[e_1, e_2, e_3]$ and $[e_3, -e_1, -e_2]$ determine the same orientation or not.

(2) What does it mean for a manifold to be orientable? Prove that $\mathbb{RP}^2$ is not orientable.

(3) Suppose f is a smooth function on $\mathbb{R}^{n+1}$ with 0 as a regular value. Show that the zero set of f is an orientable submanifold of $\mathbb{R}^{n+1}$.

Problem 5. Let X be the vector field on $\mathbb{R}^3$ such that its flow Φ_t is the rotation which fixes x-axis and rotates in the yz-plane counterclockwise by t in radians. Similarly, Y is the vector field that generates rotations around the y-axis.

(1) Find X, Y explicitly.
(2) Compute $[X, Y]$.
(3) Determine the flow generated by $[X, Y]$.

Problem 6. On $\mathbb{R}^3 - \{0\}$ consider the differential form

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

(1) Show that ω is closed.
(2) Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3 \backslash \{0\}$ be the inclusion map. Show that $\iota^* \omega$ is an orientation form on $\mathbb{S}^2$, i.e. it is nowhere zero.
(3) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary s.t. 0 is contained in the interior of Ω. Show that $\int_{\partial\Omega} \omega = 4\pi$.