Geometry Qualifying Examination

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Instructions: Solve 4 out of the 5 problems. You must justify all your claims either by direct arguments or by referring to well known and basic theorems. Indicate clearly which four problems you would like us to grade.

Problem 1.

(1) Explain what it is an orientation form on a smooth manifold M of dimension n.

(2) Consider the 2-form $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$ on $\mathbb{R}^3$. Let $i : \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion of the standard sphere. Show that $i^*\omega$ is an orientation form on $\mathbb{S}^2$.

(3) Evaluate $\int_{\mathbb{S}^2_+} \omega$, where $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 : z \geq 0\}$ is the upper hemisphere with the orientation defined by $i^*\omega$.

Problem 2. Let M be a smooth manifold of dimension n and $\mathfrak{X}(M)$ the space of vector fields on M. Let $Z \in \mathfrak{X}(M)$ and θ a 2-form on M. Define $\Phi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ by

$$\Phi(X, Y) = Z(\theta(X, Y)) - \theta([Z, X], Y) - \theta(X, [Z, Y]).$$

Prove that there is a unique 2-form ω s.t. $\Phi(X, Y) = \omega(X, Y)$. (In other words, Φ defines a 2-form.)

Problem 3. Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a smooth map $F : M \to \partial M$ s.t. $F|_{\partial M}$ is the identity map of ∂M.

Problem 4. Let $M(n, \mathbb{R})$ be the space of $n \times n$ real matrices. It is naturally identified with $\mathbb{R}^{n^2}$. Let $O(n) = \{A \in M(n, \mathbb{R}) : AA^t = I_n\}$.

(1) Show that $O(n)$ is a smooth submanifold of $M(n, \mathbb{R})$.

(2) Describe the tangent space of $O(n)$ at the identity I_n

(3) Is $O(n)$ compact? Prove your claim.

Problem 5. On $\mathbb{R}^{2n}$ with coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ consider the two-form:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

If $H \in C^\infty(\mathbb{R}^{2n})$ define a vector field X_H by $X_H \lrcorner\, \omega = dH$.

(1) Find an explicit expression for X_H in terms of partial derivatives of the function H.
(2) Show that H is constant along an integral curve of X_H.
(3) For two functons $f, g \in C^\infty(\mathbb{R}^{2n})$ compute $[X_f, X_g]$.