Geometry Qualifying Examination

Jan 4, 2024

Instructions: Solve 5 out of the 6 problems. You must justify all your claims either by direct arguments or by referring to well known and basic theorems. Indicate clearly which 5 problems you would like us to grade.

Problem 1. Let M be a nonempty smooth n-manifold, and suppose $n \geq 1$.

(1) Show that the vector space $C^{\infty}(M)$ is infinite-dimensional.

(2) Suppose $A, B \subset M$ are disjoint closed subsets. Prove that there exists $f \in C^{\infty}(M)$ s.t. $f|_A \equiv 0$ and $f|_B \equiv 1$.

(3) Suppose N is another smooth manifold and $F : M \to N$ is a continuous map. Prove that F is smooth if $F^*(C^{\infty}(N)) \subset C^{\infty}(M)$.

Problem 2. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x, y, z) = x^2 + xy - 2y^2 + z^4$.

(1) Prove that the level set $\Sigma := F^{-1}\{1\}$ is a submanifold of $\mathbb{R}^3$.

(2) Is Σ compact? Prove your claim.

(3) Find the critical points of the function $f(x, y, z) = x$ on Σ.

Problem 3. Let M be a smooth n-manifold.

(1) What is a smooth vector filed on M?

(2) Suppose (U, x) and (V, y) are two charts on M with $U \cap V \neq \emptyset$. A vector field X on M has the local expression $\sum_i a^i \frac{\partial}{\partial x^i}$ on U and $\sum_i b^i \frac{\partial}{\partial y^i}$on V. Find a formula for a^i in terms of b^j on $U \cap V$.

(3) On the local chart $\left(\mathbb{S}^2 \backslash \{N\}, x\right)$by the stereographic projection from the north pole (recall $x = \frac{(\xi^1, \xi^2)}{1-\xi^3}$), $\frac{\partial}{\partial x^1}$ defines a vector field on $\mathbb{S}^2 \backslash \{N\}$. Is it the local expression of a smooth vector field X on $\mathbb{S}^2$? If yes, what is the value of X at the north pole? Justify your answers.

Problem 4. Let X be a smooth vector field on a smooth manifold M.

(1) If $c : I \to M$ is a nonconstant integral curve, show that it is an immersion.

(2) If an integral curve $c : \mathbb{R} \to M$ is not injective, show that it is periodic, i.e. there exists $T > 0$ s.t. $c(t + T) = c(t)$, for all $t \in \mathbb{R}$.

(3) For $X = -x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$, find the maximal integral curve starting at the point $(1, 0)$.

Problem 5. Let M be a smooth n-manifold.

(1) What is a Riemannian metric on M?

(2) Suppose (U, x) and (V, y) are two charts on M with $U \cap V \neq \emptyset$. A Riemannian metric g on M has the local expression $\sum_{i,j} g_{ij} dx^i \otimes dx^j$ on U and $\sum_{i,j} \widetilde{g}_{ij} dy^i \otimes dy^j$ on V. Find a formula relating $[g_{ij}]$ and $[\widetilde{g}_{ij}]$ on $U \cap V$.

(3) Suppose M is oriented. Show that the n-form

$$\Omega = \sqrt{\det [g_{ij}]} dx^1 \wedge \cdots \wedge dx^n$$

on each positively oriented local chart (U, x) is globally defined on M.

Problem 6. On $\mathbb{R}^n \backslash \{0\}$ consider the differential $(n-1)$-form

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

(1) Show that ω is closed.

(2) Let $\iota : \mathbb{S}^{n-1} \to \mathbb{R}^n \backslash \{0\}$ be the inclusion map. Show that $\iota^* \omega$ is an orientation form on $\mathbb{S}^{n-1}$, i.e. it is nowhere zero.

(3) Is ω exact? Justify your claim.