Geometry Qualifying Exam

August 2025

DIRECTIONS: **Do Problems 1–4**, and then **do two** of the remaining Problems 5–7.

Make clear which problem you do **NOT** want graded. *All manifolds, functions, vector fields, etc. are assumed to be smooth.*

Problem 1. Let S be the subset of $\mathbb{R}^4$ defined by the equations

$$\begin{cases} x^4 + y^4 + z^4 + 3w^6 = 5 \\ x + y + z = 0 \end{cases}$$

(a) Prove that S is a submanifold of $\mathbb{R}^4$.

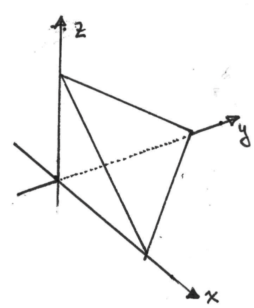
(b) What is $\dim S$? Is S compact?

Problem 2. Let $T \subset \mathbb{R}^3$ be the solid tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$. The orientation form $dx \wedge dy \wedge dz$ on T induces "outward normal" orientations on each of its faces. Using these orientations, evaluate

$$I = \int_S dx \wedge dy + dy \wedge dz$$

where $S = S_1 \cup S_2 \cup S_3$ is the union of

$S_1 =$ face in the xy-plane
$S_2 =$ face in the xz-plane
$S_3 =$ face in the yz-plane



Problem 3. Let $\Omega^p(M)$ denote the set of smooth p-forms on a smooth manifold M.

(a) The exterior derivative d is a linear map $d : \Omega^p(M) \to \Omega^{p+1}(M)$ for each $p \geq 0$ such that:
Fill in the blanks for (i)–(iii)

(i) $d^2 =$ ____

(ii) For $f \in C^\infty(M)$, the 1-form df is defined by $df(X) =$ ______ for all vector fields X.

(iii) $d(\omega \wedge \eta) =$ ________________________________ $\forall\, \omega \in \Omega^p(M),\ \eta \in \Omega^q(M)$.

(iv) d is *local*: for each $p \in M$, the value of $d\omega$ at p depends only on the restriction of ω to an arbitrarily small neighborhood of p.

(b) Show that Properties (i)–(iv) determine $d\omega$ for a 1-form ω. *Hint: write ω in local coordinates.*

(c) Use induction to prove that properties (i)–(iv) uniquely determine $d\omega$ for all p-forms ω.

Problem 4. This problem is about the definition of vector fields as derivations. Complete the definitions:

(a) A *vector field* is a *derivation*, i.e. linear map $X : C^\infty(M) \to C^\infty(M)$ such that ____

(b) The *bracket* $[X, Y]$ of vector fields X and Y is defined by ____

(c) Prove that your answer to (b) is a vector field.

(d) For a diffeomorphism $F : M \to M$, the *pushforward* of X is the vector field F_*X defined by ____

One can show (you don't have to) that

$$F_*[X, Y] = [F_*X, F_*Y] \tag{1}$$

(e) Let F_t be the flow of a vector field Z. Replace F by F_t in (1) and differentiate with respect to t at $t = 0$ to obtain a formula involving Lie derivatives.

(f) Relate your formula in (e) to the Jacobi Identity.

Do **TWO of the remaining three problems.** Make clear which ones you chose.

Problem 5. Let $f : M \to N$ be a submersion whose image is all of N. Prove that the pullback map $f^* : \Omega^p(N) \to \Omega^p(M)$ is an injection.

Problem 6. Let $f : S^2 \to M$ be a smooth map, where S^2 is regarded as the boundary of the unit ball B^3 in $\mathbb{R}^3$. Suppose that there is a DeRham cohomology class $[\omega] \in H^2(M)$ with

$$f^*[\omega] \neq 0 \qquad \text{in } \; H^2(S^2).$$

Prove that f does not extend to a map $f : B^3 \to M$.

You may use the fact that integration of 2-forms defines an isomorphism $H^2(S^2) \xrightarrow{\cong} \mathbb{R}$.

Problem 7. Let M be an n-dimensional manifold with boundary, and $f : \partial M \to \mathbb{R}$ a function on the boundary. Show that f extends to a smooth function on M, as follows.

(a) (Local extension). Fix $p \in \partial M$. Show that there is a neighborhood U_p of p in M and a map $F_p : U_p \to \mathbb{R}$ with $F|_{\partial M \cap U_p} = f$.

(b) (Global extension). Use (a) and a partition of unity to show that there is a smooth map $F : M \to \mathbb{R}$ such that $F|_{\partial M} = f$.