Geometry Qualifying Exam

January 2025

Do all 6 problems (there is a choice in Problem 4). *All manifolds, functions, vector fields, etc. are assumed to be smooth.*

Problem 1. Consider the map $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by $f(x, y, z, w) = (x^4 + y^3, zw)$. Let $M = f^{-1}(1, 1)$.

(a) Show that M is a smooth manifold and give its dimension.

(b) Describe the tangent space T_pM as a subspace of $T_p\mathbb{R}^4$ for the point $p = (x, y, z, w) = (1, 0, 1, 0)$.

(c) For which values of a and b is $f^{-1}(a, b)$ not a regular submanifold of $\mathbb{R}^4$?

Problem 2. Consider the surface $S = \{(x, y, z) \mid y = x^2+z^2, y \leq 4\}$, oriented by the orientation form $dz \wedge dx$. Evaluate:

$$\int_S z\, dx \wedge dy$$

It may help to know that $\int_0^{2\pi} \sin^2\theta\, d\theta = \int_0^{2\pi} \cos^2\theta\, d\theta = \pi$.

Problem 3. Let I denote the identity element in the Lie group $GL(n, \mathbb{R}) = \{\text{invertible } n \times n \text{ real matrices}\}$. Consider the map $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ defined by $f(A) = A^2$.

(a) Explain why $GL(n, \mathbb{R})$ is a manifold.

(b) Compute the differential $(df)_A$ at a point $A \in GL(n, \mathbb{R})$.

(c) Prove that there are neighborhoods U, V of I in $GL(n, \mathbb{R})$ such that for every matrix $B \in V$ there is a unique matrix $A \in U$ with $A^2 = B$.

Problem 4. Do ONE of the following two (somewhat shorter) problems.

4A. Let $f : M \to N$ be a smooth map from a manifold M without boundary ($\partial M = \varnothing$) to a manifold N with $\partial N \neq \varnothing$. Show that if $p \in M$ is a point such that df_p is surjective, then $f(p) \in \operatorname{int} N$.

4B. Let ω be a p-form on a manifold M. Suppose that there is a smooth positive function f on M such that $f\omega$ is exact.

(a) Prove that there is a 1-form θ such that $d\omega = \theta \wedge \omega$.

(b) In the case where ω is a 1-form, show that $d\omega \wedge \omega = 0$.

Problem 5. Let X and Y be a vector fields on a manifold M, and let $\omega \in \Omega^*(M)$ be a differential form on M.

(a) State the definition of the Lie derivative $\mathcal{L}_X\omega$ in terms of flows.

(b) Use your definition to prove that $\mathcal{L}_X$ is a derivation on $\Omega^*(M)$.

(c) Use your definition to prove that $\mathcal{L}_X$ commutes with d.

(d) In the case where ω is a 1-form, compute $\mathcal{L}_X(\omega(Y))$. Then use Cartan's formula to show that

$$(d\omega)(X,Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y]).$$

Problem 6. A p-form ω on a manifold M has compact support if $\{x \in M \,|\, \omega(x) \neq 0\}$ is compact. Let $\Omega^p_{cpt}(M)$ denote the set of such forms. The *deRham cohomology groups with compact support* are defined by

$$H^p_{cpt}(M) = \frac{\{\omega \in \Omega^p_{cpt}(M) |\ d\omega = 0\}}{\{d\eta |\ \eta \in \Omega^{p-1}_{cpt}(M)\}}.$$

Consider the case $M = \mathbb{R}^n$.

(a) Show that $H^0_{cpt}(\mathbb{R}^n) = 0$.

(b) Construct a closed n-form ω on $\mathbb{R}^n$ with compact support and $\int_{\mathbb{R}^n} \omega \neq 0$.

(c) Use (b) to show that $H^n_{cpt}(\mathbb{R}^n) \neq 0$.