TOPOLOGY QUALIFYING EXAM – JANUARY 2026

Instructions: Solve any FOUR out of the five problems. If you work on all five problems, your top four problem scores will count towards your exam score.

NOTE: No books, notes, or resources of any kind may be used during the exam.

All work must be shown, and all answers justified appropriately.

1. The fundamental group of a (based) space X can also be described as the (based) homotopy classes of maps from the circle S^1 into X. Analogously, one can define groups $\pi_n(X)$ as the group of (based) homotopy classes of maps from S^n into X.

(a) Show that every map $f : S^2 \to S^1$ is nullhomotopic. This implies that the group $\pi_2(S^1) = 0$.

(b) Prove that not every map $g : S^2 \to S^2$ is nullhomotopic. This implies that the group $\pi_2(S^2)$ is nonzero.

2. (a) Explain how to construct a CW-complex X with homology groups as follows:

$H_0(X) = \mathbb{Z}$
$H_1(X) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$
$H_2(X) = \mathbb{Z}/3\mathbb{Z}$
$H_n(X) = 0$ for all $n \geq 3$.

(b) For the space X you constructed in part (a), compute the cohomology groups $H^i(X; \mathbb{Z}/3\mathbb{Z})$.

(c) For a finitely generated abelian group G, prove that $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

3. (a) Describe the universal cover of $\mathbb{R}P^2$.

(b) Is there a covering space W of $\mathbb{R}P^2$ such that $\pi_1(W) \cong \mathbb{Z}$?

(c) Let X be a path-connected, locally path-connected, and semi-locally simply connected space. Prove that the universal cover of X is unique up to homeomorphism.

4. (a) Give an example of two spaces that are homotopy equivalent but not homeomorphic. Justify your answer.

(b) Give an example of two spaces X and Y such that $H_i(X) \cong H_i(Y)$ for all i, but X and Y are not homotopy equivalent. Justify your answer.

5. Let X be the space obtained from the torus $T^2 = S^1 \times S^1$ and the unit disk D^2 by identifying the circle $S^1 \times pt$ with ∂D^2 by a homeomorphism (see picture).

(a) Calculate $\pi_1(X)$.

(b) Calculate the homology groups of X.