1. (15 points) Let $A, B \in \mathbb{C}^{m\times m}$ be arbitrary matrices. Show that

$$||AB||_F \leq ||A||_2||B||_F,$$

where $||.||_2$ and $||.||_F$ denote the 2-norm and Frobenius norm, respectively.

2. (15 points) Let $A \in \mathbb{C}^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

3. (15 points) Answer the following questions about properties of matrices.

- Let $A, B \in \mathbb{C}^{n\times n}$, with A nonsingular and $||A^{-1}||||B|| = q < 1$. Show that $A + B$ is nonsingular, i.e., invertible.
- Let $A \in \mathbb{C}^{n\times n}$ be skew-Hermitian, i.e., $A^* = -A$. Show that the matrix $I + A$ is nonsingular and that $(I + A)^{-1}(I - A)$ is unitary.

4. (15 points) Let $P \in \mathbb{C}^{m\times m}$ be a projector. Show P is orthogonal if and only if $P = P^*$.

5. (10 points) Suppose A is an invertible 202×202 matrix with $||A||_2 = 100$ and $||A||_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.

6. (15 points) Prove Gerschgorin's disk theorem: Let $A \in \mathbb{C}^{m\times m}$ with entries $\{a_{ij}\}$ and let $r_i = \sum_{j\neq i} |a_{ij}|$. Let D_i be the closed disk in $\mathbb{C}$ with center a_{ii} and radius r_i. If λ is an eigenvalue of A, then $\lambda \in \cup_i D_i$; in other words, λ lies within at least one of the disks D_i.

7. (15 points) Suppose the Arnoldi iteration is executed for a particular $m \times m$ matrix A and vector b until at some step n, an entry $h_{n+1,n} = 0$ is encountered.

- Show how the equality $AQ_n = Q_{n+1}\tilde{H}_n$ can be simplified in this case. Recall that Q_n is an $m \times n$ matrix whose columns are the first n columns of Q in $AQ = QH$, where H is a Hessenberg matrix. In addition, $\tilde{H}_n$ is the $(n+1) \times n$ upper-left section of H.
- Show that $\mathcal{K}_n$ is an invariant subspace of A, i.e. $A\mathcal{K}_n \subseteq \mathcal{K}_n$, where $\mathcal{K}_n$ is the n^{th} Krylov subspace, which is the span of $\{b, Ab, ..A^{n-1}b\}$?