1. (15 points) Let $A \in \mathbf{C}^{m\times n}, m \geq n$, with linearly independent columns:

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n].$$

What are the eigenvalues and eigenvectors of the projection matrix

$$P = I - A(A^*A)^{-1}A^*?$$

2. (10 points) Let $A, B \in C^{m \times m}$ be arbitrary matrices. Show that

$$\|AB\|_F \leq \|A\|_2 \|B\|_F,$$

where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the 2-norm and Frobenius norm, respectively.

3. (15 points) Let $A \in R^{n\times n}$ be a symmetrical positive definite matrix. Show that the following holds:

$$\|\mathbf{x}\|_2^2 \leq \|A^{-1}\|_2 \|\mathbf{x}\|_A^2$$

where $\mathbf{x} \in R^n$ is arbitrary, and $\|\mathbf{x}\|_A^2 =< A\mathbf{x}, \mathbf{x} >$; here $< \cdot, \cdot >$ is the Euclidean inner product.

4. (15 points) Show that if $A \in \mathbf{R}^{m\times m}$ is symmetric and positive definite, then solving the linear system $A\mathbf{x} = \mathbf{b}$ amounts to computing

$$\mathbf{x} = \sum_{i=1}^{m} \frac{c_i}{\lambda_i}\mathbf{v}_i,$$

where λ_i are the eigenvalues of A and $\mathbf{v}_i$ are the corresponding eigenvectors, and c_i are some constants determined by $\mathbf{b}$ and $\mathbf{v}_i$.

5. (15 points) Let $A \in C^{2n\times 2n}$ and $B \in C^{n\times n}$, and let I be the $n \times n$ identity matrix. Let

$$A = \begin{bmatrix} I & B \\ B^H & I \end{bmatrix}$$

with $\|B\|_2 < 1$, where B^H is the Hermitian conjugate of B. Show that

$$\|A\|_2 \|A^{-1}\|_2 = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

6. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (5 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (5 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (5 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.

7. Let $A \in \mathcal{R}^{m \times n}$, $\text{rank}(A) = r$, and $\mathbf{b} \in \mathcal{R}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{R}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{R}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{R}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v} \in \mathcal{R}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (5 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.