Please do each of the following problems, providing concise answers with justification. The total number of points is 100.

1. (15 points) Let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$.

2. (15 points) Let A be an $n \times n$ matrix with $\|A\| < 1$, where $\|\cdot\|$ is an induced matrix norm.

(a) Show that the matrix $I + A$ is nonsingular.

(b) Show that

$$\|(I + A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

3. (15 points) Let $A \in \mathbf{C}^{m\times n}, m \geq n$, with linearly independent columns:

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n].$$

What are the eigenvalues and eigenvectors of the projection matrix

$$P = I - A(A^*A)^{-1}A^*?$$

4. (10 points) Suppose A is an invertible 202×202 matrix with $\|A\|_2 = 100$ and $\|A\|_F = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa(A)$.

5. (15 points) Prove **Gershgorin's Disk Theorem**:

Let $A \in \mathbb{C}^{m\times m}$ with entries $\{a_{ij}\}$. Let $r_i = \sum_{j\neq i} |a_{ij}|$. Let D_i be the closed disk in the complex plane centered at a_{ii} with radius r_i.

Prove that if λ is an eigenvalue of A, then

$$\lambda \in \bigcup_i D_i.$$

In other words, prove that every eigenvalue lies within at least one of the disks D_i.

6. (15 points) Let

$$B = \begin{pmatrix} 3 & -6 & 41/5 \\ 0 & 1 & 1 \\ 4 & -8 & 63/5 \end{pmatrix}.$$

(a) Find the QR decomposition of B using the Gram-Schmidt orthogonalization process.

(b) Write down the unitary Householder reflector matrix that you should multiply against B in order to zero out all but the first entry of its first column.

7. (15 points) Let $\mathbf{x}_j \in \mathcal{R}^m$ be the j-th column of $X \in \mathcal{R}^{m\times n}$ be given. Let $\mathbf{y} \in \mathcal{R}^m$ and $\lambda > 0$ also be given. Given a vector $\mathbf{w} \in \mathcal{R}^n$, define the following function

$$J(\mathbf{w}) = \|X\mathbf{w} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_1,$$

where $\|\cdot\|_2$ and $\|\cdot\|_1$ denote the 2- and 1-norm in $\mathcal{R}^n$, respectively. Letting the i-th component w_i of $\mathbf{w}$ vary and the other components of $\mathbf{w}$ be fixed, consider the following one-variable minimization problem reduced from $J(\mathbf{w})$:

$$\begin{aligned} \min_{w_i} f(w_i) &\equiv \min_{w_i} \|\sum_{j=1}^{n} w_j \mathbf{x}_j - \mathbf{y}\|_2^2 + \lambda|w_i| + \lambda\sum_{j\neq i}|w_j| \\ &= \min_{w_i} \|w_i\mathbf{x}_i + \mathbf{r}\|_2^2 + \lambda|w_i| + C \\ &= \min_{w_i} \sum_{j=1}^{m}(w_i x_{ji} + r_j)^2 + \lambda|w_i| + C, \end{aligned} \tag{2}$$

where $\mathbf{r} \equiv \sum_{j\neq i} w_j\mathbf{x}_j - \mathbf{y}$ is in $\mathcal{R}^m$ with $\mathbf{r} = (r_k)_{m\times 1}$, and $C = \lambda\sum_{j\neq i}|w_j|$. Show that the optimal solution w_i^* for the minimization problem (2) is given by

$$w_i^* = \begin{cases} 0 & \text{if} \quad |a| \leq \lambda, \\ \frac{-\lambda+a}{b} & \text{if} \quad \frac{-\lambda+a}{b} > 0, \\ \frac{\lambda+a}{b} & \text{if} \quad \frac{\lambda+a}{b} < 0, \end{cases}$$

where $a = -\sum_{j=1}^{m} 2x_{ji}r_j$ and $b = \sum_{j=1}^{m} 2x_{ji}^2$.

Extra page 1.

Extra page 2.