1. (10 points) Let $A, B \in C^{m\times m}$ be arbitrary matrices. Show that

$$\|AB\|_F \leq \|A\|_2 \|B\|_F,$$

where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the 2-norm and Frobenius norm, respectively.

2. (10 points) Let $A \in R^{n\times n}$ be a symmetrical positive definite matrix. Show that the following holds:

$$\|\mathbf{x}\|_2^2 \leq \|A^{-1}\|_2 \|\mathbf{x}\|_A^2$$

where $\mathbf{x} \in R^n$ is arbitrary, and $\|\mathbf{x}\|_A^2 =< A\mathbf{x}, \mathbf{x} >$; here $< \cdot, \cdot >$ is the Euclidean inner product.

3. (15 points) Show that if $A \in \mathbf{R}^{m\times m}$ is symmetric and positive definite, then solving the linear system $A\mathbf{x} = \mathbf{b}$ amounts to computing

$$\mathbf{x} = \sum_{i=1}^{m} \frac{c_i}{\lambda_i}\mathbf{v}_i,$$

where λ_i are the eigenvalues of A and $\mathbf{v}_i$ are the corresponding eigenvectors, and c_i are some constants determined by $\mathbf{b}$ and $\mathbf{v}_i$.

4. (10 points) Let $A \in C^{m\times m}$, and let a_j be its jth column. Prove the following inequality:

$$|\det(A)| \leq \prod_{j=1}^{m} \|a_j\|_2.$$

5. Suppose $X \in R^{p\times n}$, $Y \in R^{p\times n}$, $\text{rank}(X) = r_x$, and $\text{rank}(Y) = r_y$. Let $X^TY = 0$. Show that the following inequalities hold:

   (a) (5 points) $r_x + r_y \leq n$;

   (b) (5 points) $\sigma_k(X+Y) \geq \max(\sigma_k(X), \sigma_k(Y))$ for $k \geq 1$, where $\sigma_k(X)$ indicates the k-th singular value of X.

6. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (5 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (5 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (5 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.

7. Let $A \in \mathcal{R}^{m\times n}$, $\text{rank}(A) = r$, and $\mathbf{b} \in \mathcal{R}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{R}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{R}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{R}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v}\in\mathcal{R}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (5 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.

8. (15 points) Let $\mathbf{x}_j \in \mathcal{R}^m$ be the j-th column of $X \in \mathcal{R}^{m\times n}$ be given. Let $\mathbf{y} \in \mathcal{R}^m$ and $\lambda > 0$ also be given. Given a vector $\mathbf{w} \in \mathcal{R}^n$, define the following function

$$J(\mathbf{w}) = \|X\mathbf{w} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_1.$$

Letting the i-th component w_i of $\mathbf{w}$ vary and the other components of $\mathbf{w}$ be fixed, consider the following one-variable minimization problem reduced from $J(\mathbf{w})$:

$$\begin{aligned}\min_{w_i} f(w_i) &\equiv \min_{w_i} \|\sum_{j=1}^{n} w_j\mathbf{x}_j - \mathbf{y}\|_2^2 + \lambda|w_i| + \lambda\sum_{j\neq i}|w_j| \\ &= \min_{w_i} \|w_i\mathbf{x}_i + \mathbf{r}\|_2^2 + \lambda|w_i| + C \\ &= \min_{w_i} \sum_{j=1}^{m}(w_i x_{ji} + r_j)^2 + \lambda|w_i| + C,\end{aligned}$$

where $\mathbf{r} \equiv \sum_{j\neq i} w_j\mathbf{x}_j - \mathbf{y}$ is in $\mathcal{R}^m$ with $\mathbf{r} = (r_k)_{m\times 1}$, and $C = \lambda\sum_{j\neq i}|w_j|$. Show that the optimal solution w_i^* is given by

$$w_i^* = \begin{cases} 0 & \text{if} \quad |a| \leq \lambda, \\ \frac{-\lambda+a}{b} & \text{if} \quad \frac{-\lambda+a}{b} > 0, \\ \frac{\lambda+a}{b} & \text{if} \quad \frac{\lambda+a}{b} < 0, \end{cases}$$

where $a = -\sum_{j=1}^{m} 2x_{ji}r_j$ and $b = \sum_{j=1}^{m} 2x_{ji}^2$.