1. (10 points) Prove that the determinant of a Householder reflector is negative one.

2. (10 points) Let $A \in C^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

3. (10 points) Let $\epsilon > 0$ be given, $k << \min(m, n)$, $A \in R^{m\times n}$, $C \in R^{m\times k}$, and $B \in R^{k\times n}$. Assume that

$$\|A - CB\| \leq \epsilon,$$

where B and C have rank k. Further suppose that A is not available, and only B and C are available. **Without forming** the product of C and B, design an efficient algorithm to compute an approximate reduced QR of A so that the following holds,

$$\|A - QR\| \leq \epsilon,$$

where Q is an orthonormal matrix and R is upper triangular.

4. (10 points) Show that if $A \in \mathcal{R}^{n\times n}$ is symmetric, then for $k = 1$ to n,

$$\lambda_k(A) = \max_{\dim(S)=k} \min_{\mathbf{0}\neq\mathbf{y}\in S} \frac{\mathbf{y}^T A\mathbf{y}}{\mathbf{y}^T\mathbf{y}},$$

where S is a subspace of $\mathcal{R}^n$, and $\lambda_k(A)$ designates the kth largest eigenvalue of A so that these eigenvalues are ordered,

$$\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A).$$

5. Let $A \in \mathcal{C}^{m\times n}$, $\text{rank}(A) = r$, and $\mathbf{b} \in \mathcal{C}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{C}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{C}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{C}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v}\in\mathcal{C}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (10 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.

6. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (5 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (5 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (10 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.

7. Consider the following integration formula,

$$u(x) = \int_0^1 G(x,y) f(y) dy, \tag{2}$$

where $f \in C[0,1]$ and $G(x,y)$ is given by

$$G(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x \\ x(1-y) & \text{if } x \le y \le 1 \end{cases} \tag{3}$$

Partition [0,1] into $n+1$ equal subintervals with mesh size $h = \frac{1}{n+1}$: $x_j = j * h$, $\hat{u}_j \approx u_j = u(x_j)$ for $0 \le j \le n+1$. We also introduce the following vector notation $U = (u_0, u_1, u_2, \cdots, u_n, u_n)^t$, and $F = (f_0, f_1, f_2, \cdots, f_n)^t$, and $\hat{U} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_n)^t$.

(a) (10 points) To evaluate the vector $\hat{U}$, we may approximate this integral formula (2) by the Riemann sum based on the above uniform partition,

$$\hat{u}_i = \sum_{j=0}^{n} G(x_i, y_j) f(y_j) h,$$

which will lead to a matrix-vector product $\hat{U} = \hat{G}F$ in terms of a matrix $\hat{G}$ defined by

$$\hat{G} = (h * G(x_i, y_j))_{0 \le i \le n, 0 \le j \le n}$$

and the vector F. Write down this matrix-vector product to obtain the vector $\hat{U}$ from the Riemann sum. Show that the complexity of this matrix-vector product is $O(n^2)$.

(b) (10 points) Based on the above uniform partition, use the structure of the Green's function G to design an $O(n)$ algorithm to compute the vector $\hat{U}$.