1. (10 points) Let $P \in \mathcal{C}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if P is an orthogonal projector.

2. Let $A \in \mathcal{R}^{n\times n}$ be nonsingular. Consider designing an iterative method to solve the following linear system,

$$A\mathbf{x} = \mathbf{b},$$

where $\mathbf{b} \in \mathcal{R}^n$ is given and $\mathbf{x} \in \mathcal{R}^n$ is unknown.

(a) (5 points) Write down the Jacobi iterative method for the above linear system.

(b) (5 points) Write down the Gauss-Seidel iterative method for the above linear system.

3. (10 points) Let $A \in \mathcal{R}^{n \times n}$ be a symmetric, positive definite matrix. Show that solving the linear system

$$A\mathbf{x} = \mathbf{b}$$

is equivalent to finding the minimizer $\mathbf{x} \in \mathcal{R}^n$ of the quadratic form,

$$\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A\mathbf{y} - \mathbf{y}^T\mathbf{b},$$

where $\mathbf{b} \in \mathcal{R}^n$ is a given vector and $\mathbf{x}$ is unknown.

4. (10 points) Given any symmetric, positive semi-definite matrix $A \in \mathcal{R}^{n \times n}$ and any two symmetric matrices $B \in \mathcal{R}^{n \times n}$ and $C \in \mathcal{R}^{n \times n}$, show that if $C - B$ is positive semi-definite, then

$$\text{trace}(AB) \leq \text{trace}(AC).$$

5. (10 points) Let $A \in C^{2n\times 2n}$, $B \in C^{n\times n}$ and I be the $n \times n$ identity matrix. Let

$$A = \begin{bmatrix} I & B \\ B^H & I \end{bmatrix}$$

with $\|B\|_2 < 1$, where B^H is the hermitian conjugate of B. Show that

$$\|A\|_2 \|A^{-1}\|_2 = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

6. Consider the following linear system,

$$A\mathbf{x} = \mathbf{r},$$

where $\mathbf{r} \in \mathcal{R}^n$ is given, $\mathbf{x} \in \mathcal{R}^n$ is unknown, and

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ \cdots & \cdots & \cdots & 0 & a_n & b_n \end{bmatrix}$$

is assumed to be strictly diagonally dominant with $a_1 = 0$ and $c_n = 0$:

$$|b_i| > |a_i| + |c_i|,\ i = 1, 2, \cdots, n.$$

(a) (10 points) Prove that the $n \times n$ tridiagonal matrix A is nonsingular.

(b) (10 points) Let A have the LU decomposition in the form of $A = LU$, where L is an lower triangular matrix, and U is an upper triangular matrix with 1's along its main diagonal. Derive the specific forms of L and U in terms of a_i, b_i and c_i, where $i = 1, 2, \cdots, n$.

(c) (10 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = \mathbf{r}$.

7. Consider the following integration formula

$$u(x) = \int_0^x G(x,y) f(y) dy \qquad \text{if} \quad 0 \le x \le 1, \tag{1}$$

where $f \in C^2[0,1]$ and $G(x,y)$ is given by

$$G(x,y) = \begin{cases} \sqrt{x^2 - y^2} & \text{if } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Partition [0,1] into n equal subintervals with mesh size $h = \frac{1}{n}$: $x_j = y_j = jh$, $f_j = f(x_j)$, $\hat{u}_j \approx u_j = u(x_j)$ for $0 \le j \le n$. We also introduce the following vector notations: $U = (u_1, u_2, \cdots, u_n)^t$, $F = (f_1, f_2, \cdots, f_n)^t$, and $\hat{U} = (\hat{u}_1, \cdots, \hat{u}_n)^t$.

(a) (10 points) To evaluate the vector $\hat{U}$, we may approximate the integral formula (1) by the Riemann sum based on the above uniform partition,

$$\hat{u}_i = \sum_{j=1}^{i} G(x_i, y_j) f(y_j) h, \quad i = 1, \cdots, n,$$

which will lead to a matrix-vector product $\hat{U} = \hat{G}F$ in terms of matrix $\hat{G}$ defined by

$$\hat{G} = (h\, G(x_i, y_j))_{1 \le i \le n,\, 1 \le j \le n}$$

and the vector F. Write down this matrix-vector product to obtain the vector $\hat{U}$ from the Riemann sum. Show that the complexity of this matrix-vector product is $O(n^2)$.

(b) (10 points) Based on the above uniform partition, use the structure of the Green's function G to design an $O(n)$ algorithm to compute the vector $\hat{U}$ with accuracy $O(h)$. (**Hint**: split the integral into two parts: $\int_0^{x-h}$ and $\int_{x-h}^{x}$ for $x > h$.)