Qualifying Exam—Numerical ODE 08/2025

Show intermediate results at all steps!

1.(10pt) For ODE $\dot{x} = f(x)$ with f satisfying Lipshcitz condition with bounded 2nd order derivative, prove the global convergence for the Euler scheme.

2.(10pt) Derive the absolute stability region for the backward Euler scheme.

3.(10pt) Consider the homogeneous ODE $\dot{u}(t) = f(u)$, derive the Local Truncation Error of the following midpoint method:

$$x_{n+1} = x_n + hf\left(x_n + \frac{1}{2}hf(x_n)\right).$$

4.(10pt) For ODE $dz/dt = f(z) \in \mathbb{R}^d$, show that the implicit mid-point method

$$z^{n+1} = z^n + \Delta t f\left(\frac{z^{n+1} + z^n}{2}\right),$$

conserves linear invariant $z(t)^T C = const.$ of the equation.

5.(10pt) Drive the modified equation of the **backward** Euler's method for following equation

$$\frac{dx}{dt} = 2x^2 - 1, \ \ x \in \mathbb{R}.$$

6.(10pt) Consider the Hamiltonian dynamics for $(q, p) \in \mathbb{R}^2$:

$$\frac{dq}{dt} = \nabla_p H(q, p), \quad \frac{dp}{dt} = -\nabla_q H(q, p)$$

Show that the verlet scheme is symplectic.