PDE I MTH 847 QUALIFYING EXAM August 23, 2024

Name:________________________ Signature:________________________

Write clearly and coherently! Show all your work! This is a closed book exam, notes, electronic devices, cell phones, etc., can not be used. You can use a 4×6 index card. Read all problems through once before beginning your work. Problem 8 is optional for extra credit.

Problem	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total:	

Problem 1. [10 points] Find the solution of the following initial value problem for the transport equation

$$u_t + b \cdot Du + cu = f \quad \text{in } \mathbf{R}^n \times (0, \infty), \ u = g \quad \text{on } \mathbf{R}^n \times \{t = 0\}.$$

Here $c \in \mathbf{R}$, $b \in \mathbf{R}^n$ are constants, $f(x,t) = e^{-2t}$, $g(x) = |x|^2$.

Problem 2. [10 points] Let $\mathbf{B}(0,1) = \{x \in \mathbf{R}^2 : \ |x| < 1\}$ be the open unit ball.

(a) Find the solution of the boundary value problem

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbf{B}(0,1) \\ u(x) &= 2x_1^2 + 2x_1 + x_2^2 & \text{on } \partial\mathbf{B}(0,1). \end{cases}.$$

(b) Find the value $u(0)$.

(c) Find $\max\{u(x),\ x \in \overline{\mathbf{B}(0,1)}\}$ and $\min\{u(x),\ x \in \overline{\mathbf{B}(0,1)}\}$.

(d) Is there a point $x \in \mathbf{B}(0,1)$ such that $u(x) = 0$?

Problem 3. [10 points] Consider the initial value problem for the $n = 3$ dimensional wave equation

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbf{R}^3 \times (0, \infty), \quad u = 0,\ u_t = h \text{ on } \mathbf{R}^3 \times \{t = 0\}.$$

The function $h \in C_c^\infty(\mathbf{R}^3)$ satisfies the conditions that $0 \leq h(x) \leq 1$ for all $x \in \mathbf{R}^3$, $h(x) = 1$ if $|x| \leq 1$ and $h(x) = 0$ if $|x| \geq 2$. Calculate the values of $u(x, t)$ at the points $(0, 0, 0, \frac{1}{4})$, $(0, 0, 0, 4)$, $(0, 7, 0, 5)$, and the value of $u_t(x, t)$ at $(0, 0, 0, \frac{1}{4})$.

Problem 4. [10 points] Solve the mixed initial-boundary value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} = u_{xx}, & x > 0,\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = 0, & x > 0, \\ u_t(0,t) = 2u_x(0,t) & t > 0, \end{cases}$$

where the function $g \in C^2(\mathbf{R}_+)$ vanishes near $x = 0$.

Problem 5. [10 points] Let U be a bounded open set in $\mathbf{R}^n$ with C^1 boundary. The time $T > 0$ is given Let $U_T = U \times (0, T]$, and $\Gamma_T = \overline{U_T} - U_T$. Consider the boundary value problem

$$\begin{cases} u_t - \Delta u + u^5 &= 0 \quad \text{in } U_T \\ u &= 0 \quad \text{on } \Gamma_T. \end{cases}$$

Let $u \in C^{2,1}(\overline{U_T})$ be a solution of this boundary value problem. Prove that $u \equiv 0$ in $\overline{U_T}$.

Problem 6. [10 points] Let U be a bounded open set in $\mathbf{R}^2$, and $a, b, c, d, e \in \mathbf{R}$. Suppose $u \in C^2(U)$ satisfies

$$au_{xx} + bu_{yy} + cu_x + du_y - (1 + e^2)u = 0$$

in U with positive constants $a > 0$, $b > 0$.

(a) Show that u cannot have a positive maximum or a negative minimum in the interior of U.

(b) Use this to show that if $u \in C^2(U) \cap C(\overline{U})$ is a solution of the boundary value problem

$$au_{xx} + bu_{yy} + cu_x + du_y - (1 + e^2)u = 0 \text{ in } U,\ \ u = 0 \text{ on } \partial U,$$

then $u = 0$.

Problem 7. [10 points] Let $f \in C(\mathbf{R}^n \times [0,\infty))$ and $g \in C(\mathbf{R}^n)$ be bounded functions.

Prove that there is a bounded solution u of the nonhomogeneous heat equation

$$u_t - \Delta u = f \text{ in } \mathbf{R}^n \times (0,\infty), \quad u(x,0) = g(x) \text{ on } \mathbf{R}^n$$

if $|f(x,t)| \leq \frac{1}{1+t^4}$ for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Use Duhamel's principle.

Problem 8. [10 points] **Problem 8 is optional for extra credit**.

Use the method of characteristics to solve the first order equation

$$(u_{x_1})^2 + 2x_2^2 u_{x_2} = 2, \quad (x_1, x_2) \in \mathbf{R}^2$$

with the condition

$$u(x_1, 1) = x_1^2, \; x_1 \in \mathbf{R}.$$