Math 847 Qualifying Examination January 2022

Instructions: You must show all necessary work to get full or partial credits. You can use a 3×5 index card. You can not use your book, cell phone, computer, or other notes. Read all problems through once carefully before beginning work.

Notation: $\mathbb{R}^n$ denotes the standard Euclidean space with $|x| = \sqrt{x_1^2 + ... + x_n^2}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$. $\Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j \partial x_j}$ stands for the Laplace operator in $\mathbb{R}^n$. Ω is used for any open, bounded, and smooth domain in $\mathbb{R}^n$ with $\partial\Omega$ as its boundary, and $\nu(x)$ is the unit out normal at $x \in \partial\Omega$. ω_n is the surface area for $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.

Problem 1 *Use the method of characteristics to find a solution $u(x,t)$ to the Burger's equation*

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R},\ t \geq 0, \\ u(x,0) = x \end{cases}$$

near the initial surface $t = 0$. Verify your answer by a direct differentiation.

Problem 2 *Let u be a non-constant harmonic function in $\mathbb{R}^n$ (i.e., $\triangle u = 0$). Let $B(r)$ be the solid ball of radius r centered at the origin in $\mathbb{R}^n$ and $\Omega(r)$ be the boundary of $B(r)$. That is,*

$$B(r) = \{x \in \mathbb{R}^n \mid |x| \leq r\},\ \ \Omega(r) = \{x \in \mathbb{R}^n \mid |x| = r\}.$$

(a) Prove that $g_1(r) = \int_{\Omega(1)} u(ry) dS_y$ is independent of r for $r \geq 0$.

*(b) Prove that $g_\infty(r) = \sup_{y \in \Omega(1)} |u(ry)|$ is a **strictly increasing** function of r for $r \geq 0$, and $\lim\limits_{r \to \infty} g_\infty(r) = \infty$.*

*(c) Prove that $g_2(r) = \int_{\Omega(1)} |u|^2(ry) dS_y$ is a **strictly increasing** function of r for $r \geq 0$.*

(d) Can you find a number $r_0 > 0$ such that $\frac{\partial u(x)}{\partial \nu} = 0$ for all $x \in \Omega(r_0)$? Briefly explain your answer.

Problem 3 *Consider the following initial value problem for the heat equation in* $\mathbb{R}^1 \times [0, \infty)$

$$u_t(x,t) = u_{xx}(x,t), \quad u(x,0) = \frac{1+\sin^2 x}{1+x^2}.$$

(a) Prove that there is one and only one solution such that $0 < u < 1$ *for all* $x \in \mathbb{R}$ *and* $t > 0$*. For this particular solution, prove that* $h(t) = \int_{\mathbb{R}} u(x,t)dx \equiv c_0 > 0$*. Don't try to find the explicit formula for* $u(x,t)$*!*

(b) Is it possible to find another solution $u(x,t)$ *to the given initial value problem such that* $u(0,2) = -1$ *? Explain your answer.*

(c) Prove that the solution u *in (a) satisfies the estimate* $u(x,t) \geq v(x,t)$ *for all* $x \in [0,1]$ *and* $t \geq 0$*, where* $v(x,t)$ *is the solution of the following initial and boundary value problem*

$$v_t(x,t) = v_{xx}(x,t), \quad v(x,0) = \frac{4x(1-x)}{1+x^2}, \ v(0,t) = v(1,t) = 0.$$

Problem 4 *Consider the initial value problem for the wave equation in* $R^3 \times R$

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz}, & (x,y,z) \in \mathbb{R}^3, \ t \in \mathbb{R} \\ u(x,y,z,0) = 0, & (x,y,z) \in \mathbb{R}^3 \\ u_t(x,y,z,0) = \frac{1}{(1+x^2+y^2+z^2)^2} & (x,y,z) \in \mathbb{R}^3. \end{cases}$$

(a) If u *is a solution to the initial value problem, find an explicit formula for* $u(0,0,0,t)$*.*

(b) Find a solution of the form $u(x,y,z,t) = V(r,t)$ *with* $r = \sqrt{x^2+y^2+z^2}$*.*

[Hint: Note that $W(r,t) = rV(r,t)$ *satisfies* $W_{rr} = W_{tt}$*).]*

(c) Check that $V(0,t) = u(0,0,0,t)$ *you obtained from Parts (a) and (b).*

(d) Can you find a solution $u(x,t)$ *such that* $u(-1,0,0,2) - u(1,0,0,2) = 1$*? Explain your answer.*

Problem 5 *Let* $B(0,r)$ *be the closed ball in* $\mathbb{R}^n$ $(n \geq 2)$ *of radius* r *centered at the origin. Assume that* $u(x)$ *is a harmonic function on* $B(0,r)$*.*

(a) What is the Poisson's formula for $u(x)$*?*

(b) Use this Poisson's formula for balls to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0),\ |x|<r$$

if u is nonnegative on $B(0,r)$

(c) Show that any harmonic function on whole $\mathbb{R}^n$ that is bounded above (or bounded below) must be a constant. (Note that it is different form the **Liouville's theorem.***)*

Problem 6 *Consider the following so-called sine-Gordon equation, which appears in differential geometry and relativistic field theory, with initial and boundary data*

$$\begin{cases} u_{tt} = \Delta u - \sin u, & x \in \Omega, t > 0, \\ u(x,0) = g(x), & x \in \Omega, \\ u_t(x,0) = h(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

(a) Set $E(t) = \int_\Omega [u_t^2 + |Du|^2 + u^2]dx$. Prove that $E'(t) \leq 2E(t)$, $t \geq 0$.

(b) Prove that the solution to the original problem is unique.