PDE I QUAL, JANUARY 2025

Instructions.

- You have until noon to solve the exam.
- Please write your solutions neatly and legibly.
- If in doubt, provide more, rather than less, details of proofs or computations. You can apply known theorems and facts from class or the book, but be sure to state the name as best you can (eg, "maximum principle," or "representation formula for solutions to the heat equation.")
- To help graders keep track of your solutions:
  - Write your solution to each question on a separate sheet of paper. (You may use several sheets per question.)
  - Label each sheet with your name and the question number.
  - Do not write on the back of sheets.
  - Paperclip solution sheets and cover sheet together at the end of the exam.

Helpful formulas.

- Fundamental solution of Laplace's equation in $\mathbb{R}^n$:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| \text{ for } n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} \text{ for } n \geq 3. \end{cases}$$

- Green's Function for the half space: $\Phi(y-x) - \Phi(y-\tilde{x})$, where $\tilde{x}$ is the reflection of x through the plane $\{x_n = 0\}$.
- Green's Function for the unit ball: $\Phi(y-x) - \Phi(|x|(y-\bar{x}))$, where $\bar{x}$ is the inversion of x through the unit sphere $\partial B(0,1)$.
- Fundamental solution of the heat equation in $\mathbb{R}^n$:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \text{ for } x \in \mathbb{R}^n \text{ and } t > 0 \\ 0 \text{ for } x \in \mathbb{R}^n \text{ and } t = 0. \end{cases}$$

- Representation formula for solutions of solution of wave equation in 1 dimension with initial data $u(x,0) = g$ and $u_t(x,0) = h$:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- Characteristic system for $F(Du, u, x) = 0$:

$$\begin{cases} \dot{p} = -D_x F - p D_z F, \\ \dot{z} = D_p F \cdot p, \\ \dot{x} = D_p F. \end{cases}$$

- Hopf-Lax formula for solution of $u_t + H(Du) = 0$ and initial condition g, where $L = H^*$:

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

- Rakine-Hugoniot condition for a shock of a solution of $u_t + F(u)_x = 0$ along a curve C with slope σ:

$$[[F(u)]] = \sigma[[u]].$$

(1) Let $U \subset \mathbb{R}^n$ be open, bounded, and with smooth boundary. Suppose $u \in C^2(\bar{U})$ is harmonic in U. Prove that

$$\int_{\partial U} \frac{\partial u}{\partial \nu} = 0.$$

(2) Let $U \subset \mathbb{R}^2$ be open, bounded, and with smooth boundary and let $f : U \to \mathbb{R}$ be smooth. Suppose u is smooth and satisfies,

$$\begin{cases} u_t = uu_x + uu_y + \Delta u & \text{in } U \times (0,T], \\ u(x,y,0) = f(x,y) & \text{for } (x,y) \in U. \end{cases}$$

Prove

$$\max_{\bar{U}\times[0,T]} u \leq \max\left\{ \max_{(\partial U)\times[0,T]} u,\ \max_{\bar{U}} f \right\}.$$

Hint: Consider $v = u - \varepsilon t$.

(3) Given a continuous and bounded function $g : [0,\infty) \to \mathbb{R}$ with $g'(0) = 0$, find a representation formula for the solution u of

$$\begin{cases} u_t - u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u_x(0,t) = 0 & \text{for all } t > 0, \\ u(x,0) = g(x) & \text{for all } x \in \mathbb{R}. \end{cases}$$

(4) Let $U \subset \mathbb{R}^n$ be bounded, open, and with smooth boundary. Consider the problem,

$$\begin{cases} u_{tt} - \Delta u + u^5 = 0 \text{ in } U \times (0,T), \\ u(x,0) = u_t(x,0) = 0 \text{ for } x \in U, \\ u(x,t) = 0 \text{ for } x \in \partial U \text{ and } t \in (0,T). \end{cases}$$

Use energy methods to prove that if $u \in C^2(\bar{U} \times [0,T])$ satisfies this problem, then $u \equiv 0$ on $U \times [0,T]$.

(5) Let $a \in \mathbb{R}$ be nonzero. Consider the initial value problem,

$$\begin{cases} u_t + au_x = u^2 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = \cos(x) & \text{on } \mathbb{R}. \end{cases}$$

(a) Use the method of characteristics to find a solution u to this initial value problem.
Hint: the general solution to $\dot{\phi} = \phi^2$ is $\phi(s) = \left(-s + \frac{1}{\phi(0)}\right)^{-1}$.

(b) What is the largest T^* such that the solution u satisfies $u \in C^2(\mathbb{R} \times (0,T^*))$?

Name.

Grading chart.

Question	Holistic grade	Points
Q1		
Q2		
Q3		
Q4		
Q5		

Total points:

Reminder from syllabus:

- **E**xcellent: essentially flawless, receives 100% of the grade
- **G**ood: minor mistakes, but generally correct, receives 90% of the grade
- **F**air: missing some key steps or arguments, but on the right track, receives 60% of the grade
- **A**ttempted: an honest, but not satisfactory attempt was made, receives 30% of the grade.
- **N**o attempts: either no work is turned in, or if the submitted work is mostly nonsense, receives 0% of the grade.