

PDE I MTH 847 QUALIFYING EXAM January 9, 2026

Identification Code: :_____

Write clearly and coherently! Show all your work! This is a closed book exam, notes, electronic devices, cell phones, etc., can not be used. You can use a 4×6 index card. Read all problems through once before beginning your work. **Problem 7 and 8 are optional for extra credit.**

Problem	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total:	

Problem 1. [10 points] (a) Find the solution of the following initial value problem for the transport equation

$$u_t + b \cdot Du + cu = f \text{ in } \mathbb{R}^n \times (0, \infty), \quad u = g \text{ on } \mathbb{R}^n \times \{t = 0\}.$$

Here $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ are constants, $f(x, t) = e^t$, $g(x) = |x|^2$.

(b) Let $f \in C(\mathbb{R}^n \times (0, \infty))$ and $g \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume that $\|f(\cdot, t)\|_\infty = \sup\{f(x, t) : x \in \mathbb{R}^n\} < \infty$ for all $t > 0$. Prove that

$$|u(x, t)| \leq e^{-ct} \|g\|_\infty + \int_0^t e^{c(s-t)} \|f(\cdot, s)\|_\infty ds$$

for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$.

Problem 2. [10 points] Let $\mathbf{B}(0, 1) = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the open unit ball.

(a) Find the solution of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbf{B}(0, 1) \\ u(x) = 8x_1^2 - 8x_1 + 4x_2^2 & \text{on } \partial\mathbf{B}(0, 1). \end{cases} \cdot$$

(b) Find the value $u(0)$.

(c) Find $\max\{u(x), x \in \overline{\mathbf{B}(0, 1)}\}$ and $\min\{u(x), x \in \overline{\mathbf{B}(0, 1)}\}$.

(d) Is there a point $x \in \mathbf{B}(0, 1)$ such that $u(x) = 0$?

Problem 3. [10 points] (a) Let a be a number, $0 < a < 1$. Consider the set

$$A = \{(x, t) \in \mathbb{R}^n \times (0, \infty) : |x| \geq 1, 0 < t < a\}.$$

Prove that the wave equation

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

has a solution $u \in C^2(\mathbb{R}^n \times (0, \infty))$ that is not identically zero, but vanishes on the set A .

(b) Consider the initial value problem for the $n = 3$ dimensional wave equation

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \quad u = 0, \quad u_t = h \text{ on } \mathbb{R}^3 \times \{t = 0\}.$$

The function $h \in C_c^\infty(\mathbb{R}^3)$ satisfies the conditions that $0 \leq h(x) \leq 1$ for all $x \in \mathbb{R}^3$, $h(x) = 1$ if $|x| \leq 1$ and $h(x) = 0$ if $|x| \geq 2$. Calculate the values of $u(x, t)$ at the points $(0, 0, 0, \frac{1}{2})$, $(0, 0, 0, 3)$, $(10, 0, 0, 8)$, and the values of $u_t(x, t)$, $u_x(x, t)$ at $(0, 0, 0, \frac{1}{2})$.

Problem 4. [10 points] Solve the mixed initial-boundary value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} = u_{xx}, & x > 0, t > 0, \\ u(x, 0) = g(x), u_t(x, 0) = 0, & x > 0, \\ u_t(0, t) - 8u_x(0, t) = 0 & t > 0, \end{cases}$$

where the function $g \in C^2(\mathbf{R}_+)$ vanishes near $x = 0$.

Problem 5. [10 points] Let U be a bounded open set in \mathbb{R}^n with C^1 boundary. The time $T > 0$ is given. Let $U_T = U \times (0, T]$, and $\Gamma_T = \overline{U_T} - U_T$. Let $q \in C(U)$ be a nonnegative function and $f \in C(U_T)$. Consider the boundary value problem

$$\begin{cases} u_t - \Delta u + (1 + q(x))u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times (0, T) \\ u = 0 & \text{on } U \times \{t = 0\}. \end{cases}$$

Prove that a solution $u \in C^{2,1}(\overline{U_T})$ of this boundary value problem satisfies the energy inequality

$$\int_U |u(x, t)|^2 dx \leq \int_0^t \int_U e^{s-t} |f(x, s)|^2 dx ds$$

for all $0 < t < T$.

Problem 6. [10 points] Use the method of characteristics to solve the first order equation

$$(u_{x_1})^2 + 2x_2^2 u_{x_2} = 1, \quad (x_1, x_2) \in \mathbb{R}^2$$

with the condition

$$u(x_1, 1) = \frac{x_1^2}{2}, \quad x_1 \in \mathbb{R}.$$

Problem 7. [10 points] **Problem 7 is optional for extra credit.**

Let $f \in C(\mathbb{R}^n \times [0, \infty))$ and $g \in C(\mathbb{R}^n)$ be bounded functions.

(1) Prove that there is a solution of the nonhomogeneous heat equation

$$u_t - \Delta u = f \text{ in } \mathbb{R}^n \times (0, \infty), \quad u(x, 0) = g(x) \text{ on } \mathbb{R}^n$$

satisfying the estimate

$$|u(x, t)| \leq \sup\{|g(y)| : y \in \mathbb{R}^n\} + t \sup\{|f(y, s)| : y \in \mathbb{R}^n, 0 \leq s \leq t\}$$

using Duhamel's principle.

(2) Prove that there is a bounded solution u if $|f(x, t)| \leq \frac{1}{1+t^2}$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$.

Problem 8. [10 points] **Problem 8 is optional for extra credit.**

Let U be a bounded open set in \mathbb{R}^n . Suppose $u \in C^2(U)$ satisfies

$$a\Delta u + b \cdot Du - cu = 0$$

in U with positive constants $a > 0$, $c > 0$, and $b \in \mathbb{R}^n$.

(a) Show that u cannot have a positive maximum or a negative minimum in the interior of U .

(b) Use this to show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of the boundary value problem

$$a\Delta u + b \cdot Du - cu = 0 \text{ in } U, \quad u = 0 \text{ on } \partial U,$$

then $u = 0$.