PDE I QUALIFYING EXAM, AUGUST 2025

Instructions.

- This exam is closed-book: no outside materials allowed.
- Please write your solutions neatly and legibly.
- If in doubt, provide more, rather than less, details of proofs or computations. You can apply known theorems, but be sure to state the name as best you can (eg, "maximum principle," or "representation formula for solutions to the heat equation.")

Helpful formulas.

- Fundamental solution of Laplace's equation in $\mathbb{R}^n$:
$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| \text{ for } n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} \text{ for } n \geq 3. \end{cases}$$
- Green's Function for the half space: $\Phi(y - x) - \Phi(y - \tilde{x})$, where $\tilde{x}$ is the reflection of x through the plane $\{x_n = 0\}$.
- Green's Function for the unit ball: $\Phi(y - x) - \Phi(|x|(y - \bar{x}))$, where $\bar{x}$ is the inversion of x through the unit sphere $\partial B(0, 1)$.
- Fundamental solution of the heat equation in $\mathbb{R}^n$:
$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \text{ for } x \in \mathbb{R}^n \text{ and } t > 0 \\ 0 \text{ for } x \in \mathbb{R}^n \text{ and } t = 0. \end{cases}$$
- Representation formula for solutions of solution of wave equation in 1 dimension with initial data $u(x, 0) = g$ and $u_t(x, 0) = h$:
$$u(x,t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy.$$
- Characteristic system for $F(Du, u, x) = 0$:
$$\begin{cases} \dot{p} = -D_x F - p D_z F, \\ \dot{z} = D_p F \cdot p, \\ \dot{x} = D_p F. \end{cases}$$
- Hopf-Lax formula for solution of $u_t + H(Du) = 0$ and initial condition g, where $L = H^*$:
$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$
- Rakine-Hugoniot condition for a shock of a solution of $u_t + F(u)_x = 0$ along a curve C with slope σ:
$$[[F(u)]] = \sigma[[u]].$$

(1) Suppose u is nonnegative and harmonic on $\mathbb{R}^n$.
   (a) Fix any $x, y \in \mathbb{R}^n$. For any $r > 0$, denote $R = r + |x - y|$. Prove
$$u(x) \leq \frac{|B_R(y)|}{|B_r(x)|} u(y).$$
   Here, $B_R(y)$ denotes the ball of radius R centered at y and $|B_R(y)|$ denotes its volume; and similarly for $B_r(x)$. Hint: notice $B_r(x) \subset B_R(y)$.
   (b) Use part (a) to prove that u must be constant.

(2) Let $U \subset \mathbb{R}^n$ be open, bounded, and have smooth boundary. Denote $U_T = U \times (0, T]$ and let ΓU_T denote the parabolic boundary of U_T. Consider a solution $u \in C^2(\overline{U_T})$ of
$$\begin{cases} u_t - \Delta u + u^3 = 0 \text{ in } U_T, \\ u = 0 \text{ on } \Gamma U_T. \end{cases}$$
Show that $e(t) := \frac{1}{2} \int_U u^2 \, dx$ is non-increasing in time and deduce that $u \equiv 0$.

(3) Write down a representation formula for a solution $u(x, t)$ of,
$$\begin{cases} u_t + b \cdot Du - \Delta u = 0 \text{ on } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x) \text{ on } \mathbb{R}^n, \end{cases}$$
where $b \in \mathbb{R}^n$ is a constant and $f \in C^2(\mathbb{R}^n)$. Hint: Make the ansatz that u is of the form $u(x, t) = e^{\alpha \cdot x + \beta t} v(x, t)$ for some v. Find α and β so that v solves the heat equation. Then use the representation formula for solutions of the heat equation to deduce the desired expression for u.

(4) Let $U \subset \mathbb{R}^n$ be bounded and open with smooth boundary. Consider the system
$$\begin{cases} u_{tt} - \Delta u = 0 \text{ on } U \times (0, T), \\ u(x, 0) = u_t(x, 0) = 0 \text{ for } x \in U, \\ u_t + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U \times [0, T], \end{cases}$$
where $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u on ∂U. Prove, using energy methods, that if $u \in C^2(\bar{U} \times [0, T])$ satisfies this system, then $u \equiv 0$ on $U \times [0, T]$. Hint: consider $e(t) := \frac{1}{2} \int_U |u_t(x, t)|^2 + |Du(x, t)|^2 \, dx$.

(5) Use the method of characteristics to find the solution u to the PDE
$$x_1 u_{x_1} + x_2 u_{x_2} = 2u$$
on the domain $D := \{(x_1, x_2) | \, x_2 > 1\} \subset \mathbb{R}^2$ that satisfies $u(x_1, 1) = g(x_1)$, where $g \in C^1(\mathbb{R})$ is given.