PDE I *(Fall Semester 2017)* ***Qualifying Exam, 2018.1.4***

Name:

Standard exam rules apply:

- You are not allowed to give or receive help from other students.
- All electronic devices must be turned **off** for the duration of the test and stowed. This includes phones, pagers, laptops, tablets, e-readers, and calculators.
- The only things allowed on your desk are:
  - Your writing implements, including also corrector fluids or erasers or similar.
  - A water bottle or other drink.
  - This booklet.
- This exam lasts from **2:00pm–4:00pm**.
- Only one student may take a bathroom break at any given time.

INSTRUCTOR USE ONLY:

Q#	pts	MAX
1A		2
1B		4
1C		4
2		6
3A		4
3B		4
4		4
5		4
6A		4
6B		4
TOTAL		40

Q1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary. In this question you will analyze the *damped wave* equation (also called the *telegraph equation*) on $(0,\infty)\times\Omega$:

$$\Box\phi = \partial_t \phi \tag{DW}$$

(recall that $\Box = -\partial_{tt}^2 + \triangle$).

A. (2pts) Let ${}^{(\partial_t)}\mathcal{J}$ be the energy-momentum current associated to ∂_t for the function ϕ, which we assume to solve (DW). Write down an expression for the divergence $\operatorname{div}{}^{(\partial_t)}\mathcal{J}$ that does not depend on *second derivatives* of ϕ. (You don't need to show work.)

B. (4pts) Suppose $\phi \in C^2([0,\infty)\times\overline{\Omega})$ is a solution to (DW), satisfying the boundary conditions

$$\phi(t,y) = 0, \quad t \in [0,\infty), y \in \partial\Omega. \tag{DB}$$

Prove that, whenever $0 \le t_1 < t_2$,

$$\int_\Omega |\partial_t\phi(t_1,x)|^2 + |\nabla\phi(t_1,x)|^2 \,\mathrm{d}x \ge \int_\Omega |\partial_t\phi(t_2,x)|^2 + |\nabla\phi(t_2,x)|^2 \,\mathrm{d}x. \tag{EI}$$

(Hint: use the energy method.)

C. (4pts) Suppose $\phi,\psi \in C^2([0,\infty)\times\overline{\Omega})$ are solutions to (DW) with boundary conditions (DB). Suppose further that $\phi(0,x) = \psi(0,x)$ and $\partial_t\phi(0,x) = \partial_t\psi(0,x)$ for every $x \in \Omega$. Prove that $\phi \equiv \psi$ on $[0,\infty)\times\overline{\Omega}$.

(space for work on Q1)

Q2. (6pts) Let Ω be a bounded open set on $\mathbb{R}^d$. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function. Prove that, if u_1 and u_2 are two C^2 solutions to the differential equation

$$-\triangle u = f(u)$$

such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$ on Ω.
(Hint: approach by contradiction. If the two were not equal, consider the open subset $\Omega_+ := \{u_1 > u_2\}$. Notice that necessarily $u_1 - u_2 = 0$ on $\partial\Omega_+$.)

(space for work on Q2)

Q3. Consider the following Hamilton-Jacobi equation:

$$\partial_t u + (\partial_x u)^3 = 0. \tag{HJ}$$

A. (4pts) Suppose $u \in C^1([0,T] \times \mathbb{R})$ is a solution to (HJ). Suppose further there exists $x_0 \in \mathbb{R}$ such that $u(0,x) = 0$ for every $x \leq x_0$. Prove that $u(t,x) = 0$ for every $t \in [0,T]$ and $x \leq x_0$.

B. (4pts) Prove that there does not exist $u \in C^1([0,\infty) \times \mathbb{R})$ solving (HJ) such that $u(0,x) = x^3$.

(space for work on Q3)

Q4. (4pts) Let Ω be a bounded open subset of $\mathbb{R}^d$ with C^1 boundary. Suppose $u \in C^2([0,T]\times\overline{\Omega})$ solves the initial-boundary value problem

$$\partial_t u - \triangle u + |\nabla u|^2 u = 0$$

with

$$\begin{cases} u(0,x) = 0, & x \in \Omega; \\ u(t,y) = 0, & t \in [0,T], y \in \partial\Omega. \end{cases}$$

Prove that $u \equiv 0$.

(space for work on Q4)

Q5. (4pts) Let Ω be an open, bounded subset of $\mathbb{R}^d$, and suppose $f : \Omega \to \mathbb{R}$ is such that $f(x) > \epsilon > 0$ for all $x \in \Omega$. Suppose $u \in C^2(\overline{\Omega})$ solves

$$-\triangle u + f u = 0.$$

Prove that

$$\max_{\overline{\Omega}} u \leq \max\left\{0, \max_{\partial\Omega} u\right\}.$$

(space for work on Q5)

Q6. Let $f \in C^0([0,\infty)\times\mathbb{R}^d)$ be bounded, non-negative, and compactly supported. Consider the initial value problem

$$\partial_t u - \triangle u = f, \tag{HI}$$

$$u(0,x) = 0 \tag{1}$$

A. (4pts) Prove that there exists a *bounded* solution of (HI).

B. (4pts) Suppose there exists some $(t_0, x_0) \in (0,\infty)\times\mathbb{R}^d$ such that $f(t_0, x_0) > 0$. Prove that $u(t,x) > 0$ for every $x \in \mathbb{R}^d$ and every $t > t_0$.

(space for work on Q6)