AUGUST 2018 QUALIFYING EXAM: PDE I

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose $u \in C^2((0,T]\times\Omega)\cap C^0([0,T]\times\overline{\Omega})$ solves

$$\partial_t u - \triangle u = \sin(\pi u)$$

and is such that $u(0,x) \geq 1$ for every $x \in \Omega$, and $u(t,x) = 1$ for every $t \in (0,T)$ and $x \in \partial\Omega$.

a) (4pts) Prove that, for every $\epsilon \in (0,1)$, the function $u > \epsilon$ on $[0,T]\times\overline{\Omega}$.

b) (4pts) Show that there is an *unique* solution satisfying $u(0,x) \equiv 1$. Write down the solution explicitly.

2. Let $h : \mathbb{R} \to \mathbb{R}$ satisfy $zh(z) \geq 0$. Let D denote the domain $\{(t,x) \in (0,T)\times(-1,1)\} \subset \mathbb{R}\times\mathbb{R}$. Suppose $\phi \in C^2(\overline{D})$ solves the initial-boundary value problem

$$-\partial^2_{tt}\phi + \partial^2_{xx}\phi = h(\partial_t\phi)$$
$$(\phi, \partial_t\phi)\Big|_{t=0} = (f, g)$$
$$\phi(t,-1) = \phi(t,1) = 0$$

a) (4pts) Let $E(t) := \int_{-1}^{1} |\partial_t\phi(t,x)|^2 + |\partial_x\phi(t,x)|^2 \, \mathrm{d}x$. Prove that $E(t)$ is decreasing in t.

b) (4pts) Suppose $f(x) = g(x) = 0$ when $x \geq 0$. Prove that $\phi(t,x) = 0$ whenever $x \geq t \geq 0$.

3. (4pts) Prove the following stronger version of Liouville's theorem:
Let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function. If there exist $D_1, D_2 > 0$ and $\epsilon \in [0,1)$ such that for every $x \in \mathbb{R}^n$

$$|u(x)| \leq D_1|x|^\epsilon + D_2$$

then u is constant.

4. Consider the Poisson equation $-\triangle u = f$ on $\mathbb{R}^n$, where $f \in C^2_c(\mathbb{R}^n)$ is given.

a) (4pts) When $n \geq 3$, prove that for every non-negative f, there exists an unique non-negative C^2 solution to the Poisson equation $-\triangle u = f$ on $\mathbb{R}^n$ with the property that $\lim_{|x|\to\infty} u(x) = 0$. (*Your emphasis should be on the* non-negativity *and* uniqueness *of the solution.*)

b) (4pts) When $n = 2$, prove that if f is non-negative and non-trivial, any C^2 solution to the Poisson equation is unbounded. (*You may use the version of Liouville's theorem stated in question 3.*)

5. (4pts) Let $h \in C^\infty(\mathbb{R})$ be such that at every $y \in \mathbb{R}$, there exists some natural number $k(y) \geq 1$ such that the kth derivative $h^{(k)}(y) \neq 0$. Prove that the only $C^1(\mathbb{R}\times\mathbb{R})$ solutions to

$$\partial_t u + h(u)\partial_x u = 0$$

are the constant solutions.

6. (4pts) Let $u \in C^4([0,T]\times\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open, bounded, and has C^1 boundary. Suppose u solves the *biharmonic heat equation*

$$\partial_t u + \triangle\triangle u = 0$$
$$u(t,x) = 0 \quad \text{on } [0,T]\times\partial\Omega$$
$$\partial_\nu u(t,x) = 0 \quad \text{on } [0,T]\times\partial\Omega$$

Show that if $u(T,x) = 0$ for all $x \in \Omega$, then $u \equiv 0$ on $[0,T]\times\Omega$.
(*Hint: letting* $E(t) = \int_\Omega |u(t,x)|^2 \, \mathrm{d}x$, *you can start by showing that the function* $t \mapsto \ln E(t)$ *is convex.*)

Date: August 23, 2018.