MTH849- AUGUST 2020 QUALIFYING EXAM

1. Instructions

There are five problems on this exam. Complete as many problems as possible. Your four highest scoring answers will be used to determine your grade on the exam. A preference in scoring will be given to complete answers to entire problems, in contrast to partial answers to possibly more problems. You are permitted one extra sheet of paper of your own construction that contains theorems, and other results from the textbook and lecture notes.

2. The exam questions

Problem 1. Assume that $B_1(0) \subset \mathbb{R}^n$ is the unit ball. For each of the following functionals, $\ell_1 - \ell_4$, you will have two answer two questions, (i) and (ii), given below.

(i) Which of the following can be extended from $C^1(\overline{B_1(0)})$ to a bounded linear functional on $H^1(B_1(0))$? Either prove why, or give an example or brief explanation showing why the functional cannot be extended.

(ii) Which of the following can be extended from $C^1(\overline{B_1(0)})$ to a bounded linear functional on $L^2(B_1(0))$? Either prove why, or give an example or brief explanation showing why the functional cannot be extended.

For time considerations, when you prove why a functional may be extended, you may just give the main ideas, without writing a complete proof; and when giving an example, just be as precise as time allows, which may only mean you give a vague line of reasoning. You can come back later and increase the details, if time permits.

For $f \in C^1(\overline{B_1})$, the linear functionals are:

(a)

$$\ell_1(f) = \int_{\partial B_1(0)} f(x)g(x)dx,$$

where $g \in L^2(\partial B_1)$ is a fixed function.

(b)

$$\ell_2(f) = f(0)$$

(c)

$$\ell_3(f) = \int_{B_{1/4}(0)} \frac{\partial f}{\partial x_1}(x)dx$$

(d)

$$\ell_4(f) = \int_{\partial B_1(0)} \frac{\partial f}{\partial \nu}(x)g(x)dx,$$

where ν is the outward normal vector to the domain, and $g \in C(\partial B_1(0))$ is a fixed function.

Problem 2. Assume each of the following equations admits at least one weak solution– defined in the appropriate sense given below– and call it u_0. For each equation, say whether or not the solution is unique. If the solution is unique, then prove why. If the solution is not unique, then give an example.

For time considerations, when you prove why a particular equation may have a unique solution, you may just give the main ideas, without writing a complete proof.

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open, connected set, with a C^1 boundary, and that $\Gamma \subset \partial\Omega$ and $|\Gamma|_{\partial\Omega} > 0$.

(1)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H_0^1(\Omega)$ satisfies

$$\forall\ v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

(2)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \Gamma \subset \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_\Gamma T v dS.$$

(3) for $c_0 > 0$,

$$\begin{cases} -\Delta u + c_0 u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \Gamma \subset \partial\Omega. \end{cases}$$

Here, a weak solution $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c_0 u v dx = \int_\Omega f v dx + \int_\Gamma T v dS.$$

(4)

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \text{no boundary condition given} & \end{cases}$$

Here, we abuse notation to say a weak "solution" $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx.$$

(This is a slight of hand, because integrating the actual equation by parts would yield a boundary term, but we are ignoring those, intentionally, but nonetheless it is not really a true solution.)

(5) for $c_0 > 0$,

$$\begin{cases} -\Delta u + c_0 u = f & \text{in } \Omega \\ \text{no boundary condition given} \end{cases}$$

Here, we abuse notation to say a weak "solution" $u \in H^1(\Omega)$ satisfies

$$\forall\ v \in H^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega c_0 uv dx = \int_\Omega fv dx.$$

(This is a slight of hand, because integrating the actual equation by parts would yield a boundary term, but we are ignoring those, intentionally, but nonetheless it is not really a true solution.)

As a hint to all parts, one may consider the equation solved in the weak sense by the difference of any two possible solutions.

Problem 3. Assume that $B_1(0) \subset \mathbb{R}^n$ is the unit ball, and that a^{ij} are uniformly elliptic coefficients. You will prove an energy estimate for weak solutions of

$$-\frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) = f + \frac{\partial}{\partial x_j} g_j \ \text{ in } B_1.$$

When u is a weak solution of the Dirichlet problem, with $u \in H_0^1(B_1)$, i.e.

$$\forall\ v \in H_0^1(B_1), \quad \int_{B_1} a^{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx = \int_{B_1} fv dx - \int_{B_1} g_j \frac{\partial v}{\partial x_j} dx,$$

prove the estimate,

$$\int_{B_1} |\nabla u|^2 \leq C\left(\int_{B_1} u^2 dx + \int_{B_1} f^2 dx + \sum_j \int_{B_1} |g_j|^2\right).$$

Problem 4. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, with a C^1 boundary and that a^{ij} is uniformly elliptic (and bounded). Prove that if f, g_j are in $L^2(\Omega)$ (for $j = 1, \dots, n$), then there exists a unique solution of

$$\begin{cases} -\frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) = f + \frac{\partial}{\partial x_j} g_j & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this context, we mean that $u \in H_0^1(\Omega)$ satisfies

$$\forall\ v \in H_0^1(\Omega), \quad \int_\Omega \frac{\partial v}{\partial x_j}(x) a^{ij}(x)\frac{\partial u}{\partial x_i}(x)dx = \int_\Omega fv dx - \int_\Omega g_j(x)\frac{\partial v}{\partial x_j}(x)dx.$$

If for some reason, the coefficients a^{ij} are a problem, then attempt a solution assuming that $a^{ij} = \delta_{ij}$, meaning (a^{ij}) is the identity matrix, and an appropriate modified score will be assessed.

Problem 5. Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded, connected, with a C^1 boundary, and that a^{ij} is uniformly elliptic. Provide conditions on f, g_j, and h so that the following equation can admit a weak solution. Furthermore, use your condition to give a definition of weak solution to this equation, and prove under your condition and definition that there exists a unique weak solution.

$$\begin{cases} -\frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right) = f + \frac{\partial}{\partial x_j} g_j & \text{in } \Omega \\ a^{ij}\frac{\partial u}{\partial x_i}\nu_j = h & \text{on } \partial\Omega \end{cases}$$

(here, ν_j is the j-th component of the normal vector ν).

If for some reason, the coefficients a^{ij} are a problem, then attempt a solution assuming that $a^{ij} = \delta_{ij}$, meaning (a^{ij}) is the identity matrix, and an appropriate modified score will be assessed.