

PDE II QUALIFICATION EXAM (AUGUST 2023)

Instructions.

- No outside resources of any kind are allowed.
- Write solutions legibly and in an organized manner.
- Write your solution to each question on a separate answer sheet (you may use multiple sheets per question), label each sheet with your qual number, question number, sheet number, and total number of sheets used. Do not write on backs of sheets.
- Problems will be graded according to the holistic grading scheme as detailed on the back of this page.

Unless otherwise specified, $\Omega \subset \mathbb{R}^n$ is a bounded, open set with C^1 boundary.

Problem 1. Fix $0 < p < \infty$ and let us write (x, y) for points in \mathbb{R}^2 . Suppose that for some $1 \leq q < \infty$ we want to prove an inequality of the form

$$\|u\|_{L^q(\mathbb{R}^2)} \leq C \left(\int_{\mathbb{R}^2} \left| \frac{\partial^3 u(x, y)}{\partial x \partial y^2} \right|^p + \left| \frac{\partial^3 u(x, y)}{\partial x^3} \right|^p \left| \frac{\partial u(x, y)}{\partial y} \right|^p dx dy \right)^{\frac{1}{p}}, \quad \forall u \in C_c^3(\mathbb{R}^2)$$

where $C > 0$ is independent of u . Using a scaling argument, prove there is only one possible value of q for each p such that the above inequality can hold, and find this value. You do not need to prove the inequality holds for that value.

Problem 2. Prove that if $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ are functions satisfying

$$-\sum_{i=1}^n (1 + \arctan(|\partial_i u|))^3 \partial_i^2 u + e^u \leq -\sum_{i=1}^n (1 + \arctan(|\partial_i v|))^3 \partial_i^2 v + e^v \text{ on } \Omega, \quad u \leq v \text{ on } \partial\Omega$$

then $u \leq v$ on all of Ω .

Problem 3. Fix a positive function $\eta \in L^2(\Omega)$ and suppose $n \geq 3$. Prove that if $q \in [2, n]$, then $\int_{\Omega} \eta |u| |v| < \infty$ for all $(u, v) \in W^{1, \frac{n}{2}}(\Omega) \times W^{1, q}(\Omega)$. (Note: you will need to treat the case $q = n$ separately).

Problem 4. For each $k \in \mathbb{N}$, let $L_k u := -\partial_j (a^{ij} \partial_i u) + \frac{1}{k} u$ where $a^{ij} = 1$ when $i = j$ and 0 when $i \neq j$. Then suppose $u_k \in H_0^1(\Omega)$ is a weak solution to the equation

$$L_k u = \frac{1}{k^2} \text{ on } \Omega, \quad u \equiv 0 \text{ on } \partial\Omega.$$

Prove that:

- (1) $\sup_k \|u_k\|_{L^2(\Omega)} < \infty$.
- (2) There is a subsequence of $\{u_k\}_{k=1}^{\infty}$ which converges in $L^2(\Omega)$ norm.

Problem 5. Let Ω be the open triangle $\{(x, y) \mid 0 < x < 1, 0 < y < 2 - 2x\} \subset \mathbb{R}^2$, define $(a^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{2} - 2x - y \end{pmatrix}$, and $Lu = -\partial_j (a^{ij} \partial_i u)$. Using Rayleigh's formula, give a nontrivial lower bound on the principal Dirichlet eigenvalue for the eigenvalue problem

$$Lu = \lambda u \text{ on } \Omega, \quad u \equiv 0 \text{ on } \partial\Omega.$$

You may give any lower bound if it is **fully justified**, but the bound must be **strictly positive** and **an explicit number** to receive credit.

Question	Holistic grade	Points
Q1		
Q2		
Q3		
Q4		
Q5		
Total		

Grading scale

- **Excellent:** essentially flawless, 100% of grade.
- **Good:** minor mistakes, but generally correct, 90% of grade.
- **Fair:** missing some key steps or arguments, but on the right track, 70% of grade.
- **Wrong approach:** uses a not-promising approach to the problem but concludes the approach in a correct manner to its logical conclusion, 50% of grade.
- **Attempted:** an honest but not satisfactory attempt, 30% of grade.
- **No attempt:** no work or mostly nonsense solution, 0% of grade.