Qualifying Exam – Partial Differential Equations II – August 2024

INSTRUCTIONS: ***The exam is closed-book***– *Notes, resources or communications are strictly prohibited*! You may use the well-known theorems or results without proof, but you must indicate such a result clearly with all necessary steps. You may also use the result of one problem to solve another problem on the exam. You must work on all problems to pass the exam.

In all the following, unless specified, Ω ***will denote a bounded open set in*** $\mathbb{R}^n$.

#1. (15 points) Let $n \geq 2$, $1 < p < n$, and let Ω be an open set containing 0 in $\mathbf{R}^n$. Prove that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_\Omega |Du(x)|^p dx \quad \forall\, u \in W_0^{1,p}(\Omega).$$

#2. (10 points) Let $B_R(a)$ denote the open ball in $\mathbb{R}^n$ with center a and radius R. Prove the inequality

$$\int_{B_R(a)\setminus \bar{B}_{R/3}(a)} (u - u_R)^2 dx \leq C_n R^2 \int_{B_R(a)\setminus \bar{B}_{R/3}(a)} |Du|^2 dx \quad \forall\, u \in H^1(B_R(a)),$$

where u_R is the average of u over $B_R(a) \setminus \bar{B}_{R/3}(a)$ and C_n is a constant independent of R and a.

#3. Let $u \in H^1_{loc}(\mathbb{R}^n)$ be a weak solution of equation $-\sum_{i,j=1}^n D^i(a_{ij}(x)D^j u) = 0$ in $B_R(0)$ for all $R > 0$, where the equation is assumed to be uniformly elliptic in all of $\mathbb{R}^n$.

(a) (10 points) Show that there exists a constant C such that for all $R > 0$ and $\lambda \in \mathbb{R}$,

$$\int_{B_{R/3}(0)} |Du|^2 dx \leq \frac{C}{R^2} \int_{B_R(0)\setminus \bar{B}_{R/3}(0)} (u - \lambda)^2 dx.$$

(b) (10 points) Assume $|Du| \in L^2(\mathbb{R}^n)$. Show that u is constant almost everywhere on $\mathbb{R}^n$.

#4. (15 points) Let $\beta \in C^1(\mathbb{R}^n)$. Show that given any f and g, the boundary value problem

$$\begin{cases} -\Delta u + \beta(Du) + u^3 = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

can have at most one solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

#5. Let $\Phi \in C^1(\bar{\Omega}; \mathbb{R}^n)$ and $c \in C(\bar{\Omega})$. Consider the operator

$$Lu = -\Delta u + \Phi \cdot Du + cu.$$

(a) (10 points) Find the formal adjoint operator L^* of L on $H_0^1(\Omega)$ in the sense that the bilinear form B^* associated with L^* satisfies $B^*[u, v] = B[v, u]$ for all $u, v \in H_0^1(\Omega)$, where B is the bilinear form associated with L.

(b) (15 points) Show that $Lu = f$ has a unique weak solution $u \in H_0^1(\Omega)$ for each $f \in L^2(\Omega)$ provided that one of the following conditions holds:

(i) $c \geq 0$ in Ω. (ii) $c \geq \operatorname{div} \Phi$ in Ω. (iii) $c \geq \frac{1}{2} \operatorname{div} \Phi$ in Ω.

#6. (15 points) Let $n \geq 2$, $1 < p < n$, $0 < q < \frac{np+p-n}{n-p}$ and $q \neq p - 1$. Show that the boundary value problem

$$\begin{cases} -\operatorname{div}(|Du|^{p-2} Du) = |u|^{q-1} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial weak solution in $W_0^{1,p}(\Omega)$ satisfying $u \geq 0$ almost everywhere in Ω.